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Caution: Falling Dice!

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Presentation of the research topic

Two fair dice are rolled together, repeatedly. A fair die is a die for which all sides have the same chance of appearing at every roll. We consider two events related to this random experiment:

A= the double $(6,6)$ *appears;*

B= the sum 7 *appears twice in a row.*

Here, we interpret "sum 7" as the sum of the results from both dice that appeared at one roll.

a) Determine the probability of event *A* happening in a single roll.

b) How may rolls are needed, on average, until we get the double $(6,6)$ for the first time (that is, the expected value of the number of rolls)?

c) Determine the probability of event *B* happening in exactly two rolls.

d) How many rolls are needed, on average, to get event *B* to happen for the first time?

Two friends, Anna and Bill, are playing a game based on rolling two fair dice repeatedly. Anna says that event *A* will happen before event *B* happens. Bill says the opposite of that. The dice are thrown repeatedly until one of the kids wins.

e) Who has a higher chance of winning the game? Determine Anna's odds of winning the game. For example, if her probability of winning is $\frac{5}{8}$, then her odds are $5:3$.

f) Make comments on the result based on the results of the previous points.

g) We have the information that the first roll of the dice was not the double $(6,6)$. What are now Anna's odds of winning the game?

Brief presentation of the conjectures and results obtained

This article presents an introduction to the study of probabilities, through a problem with 7 tasks. In order to approach the problem, we explain in the beginning of the article some basic properties and definitions about probability, expected values and discrete random variables. The problem requires us to find the probability that some events happen when rolling dice, or their mathematical expectancy. Our problem impresses because, although it can be seen as a children's game, it contains a lot of useful information for the field of statistics.

The text of the article

1. Definitions and Properties

1.1. Classical Definition of Probability. If a random experiment (process with an uncertain outcome) can result in a positive integer *n* of mutually exclusive and equally likely outcomes, and if n_A of these outcomes have an attribute *A*, then the probability of *A* is the fraction $\frac{n_A}{n}$.

1.2. Definition of incompatible events. Two events that cannot happen simultaneously are called *incompatible* or *mutually exclusive*.

1.3. Axiomatic Definition of Probability. If we do a certain experiment, which has a finite sample space Ω, we define the probability as a function that associates a certain probability, $p(A)$ with every event A, satisfying the following properties:

1.3.1. The probability of any event *A* is positive or zero. Namely $p(A) \geq 0$. The probability measures, in a certain way, the chance of event *A* to happen: the smaller the probability, the less chances for *A* to happen.

1.3.2. The probability of the sure event is 1. Namely $p(\Omega)=1$. And so, the probability is always greater than 0 and smaller than 1: probability 0 means that there is no probability for it to happen (it is an impossible event), and probability 1 means that it will always happen (it is a sure event).

1.3.3. The probability of the union of every set of two by two incompatible events is the sum of the probabilities of the events. That is, if we have, for example events *A*, *B*, *C*, and these are two by two incompatible, then $p(A \cup B \cup C) = p(A) + p(B) + p(C)$.

1.4. Main Properties of Probability.

1.4.1. The probabilities of complementary events add up to 1:

$$p(A) + p(\bar{A}) = 1.$$

Often we will use this property to calculate the probability of the complementary set.

This property, which turns out to be very useful, can be generalized. If we have three or more events, two by two incompatible, and such that their union is the whole space, that is to say, *A*, *B*, *C*, two by two incompatible so that $A \cup B \cup C = \Omega$, then $p(A)+p(B)+p(C)=1$. We say in this case that *A*, *B*, *C* form *a complete system of events*. Let's observe that whenever we express Ω as a set of elementary events, in fact we are giving a complete system of events.

1.4.2. The probability of the impossible event is zero:

$$p(\varnothing)=0.$$

1.4.3. If $A \subset B$, then $p(A) \le p(B)$.

The notation "if $A \subset B$" reads "if the event *A* is included in event *B*", that is to say if all the possible results that satisfy *A* also satisfy *B*.

1.4.4. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.

1.5. Definition of Discrete Random Variables. A *random variable* is a function that assigns a numerical value to each possible outcome of a probabilistic event. A *discrete random variable* can take only distinct, separate values.

A probability distribution for a discrete random variable *X* consists of:

- All its possible values $x_1, x_2, \ldots, x_n, \ldots$
- Corresponding probabilities $p_1, p_2, \ldots, p_n, \ldots$

with the interpretation that $p(X=x_1)=p_1,\ p(X=x_2)=p_2, \ldots,\ p(X=x_n)=p_n, \ldots$. Each $p_i \ge 0$ and $p_1+p_2+\ldots+p_n+\ldots=1$.

1.6. Definition of Expected Value. The expected valuable of a discrete random variable *X* is

$$E(X)=\sum_{i=1}^{n} p(X=x_i)\cdot x_i = \sum_{i=1}^{n} p_i \cdot x_i, \text{ for a finite number of possible values or}$$

$$E(X)=\sum_{i=1}^{\infty} p(X=x_i)\cdot x_i = \sum_{i=1}^{\infty} p_i \cdot x_i, \text{ for a countable infinite number of possible values.}$$

2. Solution

a) Let *p* be the probability that event *A* happens in a single roll and let p' be the probability of getting a 6 when you roll a die once. We denote this event by *C*.

For event A to happen, both dice must show 6 when rolled once. So, event C must happen once for each of the two dice. This means that $p = p' \cdot p'$.

When rolling a die, the probability of number 6 appearing is $\frac{1}{6}$, i.e., $p' = \frac{1}{6}$.

Therefore, the probability of event A happening in a single roll is

$$p = p' \cdot p' = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

b) **Method I**

We denote by x the number of rolls necessary, on average, to get the double $(6,6)$ for the first time.

If we get $(6,6)$ on the first roll, then we would only need one roll. The probability of this happening is, according to the previous point, $\frac{1}{36}$.

In the opposite case, with a probability of $\frac{35}{36} = 1 - \frac{1}{36}$, if we do not get the double $(6,6)$ on the first roll, we would need $x+1$ rolls.

So, we have $x = \frac{1}{36} \cdot 1 + \frac{35}{36} \cdot (x+1)$, from which we get that $x = 36$.

Method II

Let X be the random variable associated to the event "A happens for the first time at roll number n".

The probability of event A happening in any one throw is, as we have established at point a), $p = \frac{1}{36}$, and the probability of event A not happening at any one throw is

$$r = p\left(\bar{A}\right) = 1 - p\left(A\right) = 1 - p = \frac{35}{36}.$$

Thus, the probability of event A happening for the first time at roll number k (where k is a positive integer) is $p \cdot r^{k-1}$. Therefore, we get that

$$X\begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ p & p \cdot r & p \cdot r^2 & \dots & p \cdot r^{k-1} & \dots \end{pmatrix}.$$

So, the expected value associated to the random variable X is

$$E(X) = \sum_{k=1}^{\infty}\left(k \cdot p \cdot r^{k-1}\right) = p \cdot \sum_{k=1}^{\infty}\left(k \cdot r^{k-1}\right).$$

We denote $S_n = \sum_{k=1}^{n}\left(k \cdot r^{k-1}\right) = 1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \dots + n \cdot r^{n-1}$. That yields

$$(1-r) \cdot S_n = (1-r)\left(1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \dots + n \cdot r^{n-1}\right) =$$

$$=1\cdot r^0+2\cdot r^1+3\cdot r^2+\ldots+n\cdot r^{n-1}-r-2\cdot r^2-3\cdot r^3-\ldots-n\cdot r^n=$$

$$=1+r+r^2+\ldots+r^{n-1}-n\cdot r^n=\frac{1-r^n}{1-r}-n\cdot r^n=\frac{1-r^n-n\cdot r^n\cdot(1-r)}{1-r}.$$

Therefore, $S_n=\frac{1-r^n-n\cdot r^n\cdot(1-r)}{(1-r)^2}=\frac{1-r^n-n\cdot r^n\cdot p}{p^2}$. Using this fact, we can write:

$$E(X)=p\cdot\lim_{n\to\infty}S_n=p\cdot\lim_{n\to\infty}\frac{1-r^n-n\cdot r^n\cdot p}{p^2}=\frac{1}{p}\cdot\lim_{n\to\infty}\left(1-r^n-n\cdot r^n\cdot p\right).$$

Because $r\in(0,1)$, we get $E(X)=\frac{1}{p}\cdot\lim_{n\to\infty}\left(1-r^n-n\cdot r^n\cdot p\right)=\frac{1}{p}\cdot(1-0-0)=\frac{1}{p}=36$.

c) Let q be the probability of event B happening in exactly two rolls and s be the probability than we get the sum 7 when rolling two dice. We denote this event by S (success).

For event B to happen, event S must happen twice in a row. That implies that $q=s\cdot s$.

When rolling two dice, we get one of 36 results from the set $\left\{(x,y)\,\middle|\,x,y=\overline{1,6}\right\}$.

The only favorable results (which have sum 7) are

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

Therefore, $s=\frac{6}{36}=\frac{1}{6}$, from where we get that the probability of event B happening in exactly two rolls is $q=s\cdot s=\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$.

d) **Method I**

We denote by y the number of rolls that are necessary, on average, until event B happens for the first time.

If we do not get sum 7 on the first roll, event that happens with the probability of $1-s=1-\frac{1}{6}=\frac{5}{6}$, then we would need $y+1$ rolls.

Else, if we get sum 7 on the first roll, but not on the second roll, then $y+2$ rolls would be needed. The probability of this event is $s\cdot(1-s)=\frac{1}{6}\cdot\frac{5}{6}=\frac{5}{36}$.

Lastly, if we get sum 7 on the first two throws, then we would need 2 rolls.

Thus,

$$y=\frac{5}{6}\cdot(y+1)+\frac{5}{36}\cdot(y+2)+\frac{1}{36}\cdot 2,$$

which implies $y=42$.

Method II

Let Y be the random variable associated to the event "B happens for the first time on roll number n". The possible values of Y are $\{2,3,4,\ldots\}$. If we denote by p_k the probability of B happening for the first time on roll k, then we have

$$Y\begin{pmatrix} 2 & 3 & \ldots & k & \ldots \\ p_2 & p_3 & \ldots & p_k & \ldots \end{pmatrix},$$

where $\sum_{k=2}^{\infty} p_k = 1$.

$s = \frac{1}{6}$ is the probability of getting sum 7 when rolling two dice, event which we denoted by S (success), and let $f = 1 - s = \frac{5}{6}$ be the probability of not getting sum 7 when rolling two dice, event which we denote by F (failure).

We observe that $p_1 = 0$, $p_2 = s^2$ and $p_3 = f \cdot s^2$. By considering all the possible outcomes of the event B in the first two rolls, we shall get a recurrence relation for p_k, where $k \geq 3$. Apart from the case $k = 2$, when we start with SS, for cases $k \geq 3$ we start either with F or with SF. That yields $p_k = f \cdot p_{k-1} + s \cdot f \cdot p_{k-2}$, for all $k \geq 3$.

The expected value of the number of rolls needed until event B happens for the first time is $E(Y) = \sum_{k=2}^{\infty} k \cdot p_k$.

For all $k \geq 3$ we have $k \cdot p_k = f \cdot (k-1+1)\, p_{k-1} + s \cdot f \cdot (k-2+2)\, p_{k-2}$. Therefore, we have that

$$\sum_{k=3}^{\infty} k \cdot p_k = f \cdot \left(\sum_{k=3}^{\infty} (k-1) \cdot p_{k-1} + \sum_{k=3}^{\infty} p_{k-1} \right) + f \cdot s \cdot \left(\sum_{k=3}^{\infty} (k-2) \cdot p_{k-2} + 2 \sum_{k=3}^{\infty} p_{k-2} \right)$$

$$\Rightarrow E(Y) - 2 \cdot p_2 = f \cdot \left(\sum_{k=2}^{\infty} k \cdot p_k + \sum_{k=2}^{\infty} p_k \right) + f \cdot s \cdot \left(\sum_{k=1}^{\infty} k \cdot p_k + 2 \sum_{k=1}^{\infty} p_k \right)$$

$$\Rightarrow E(Y) - 2 \cdot s^2 = f \cdot \left(E(Y) + 1 \right) + f \cdot s \cdot \left(E(Y) + 2 \right)$$

$$\Rightarrow E(Y) \cdot (1 - f - f \cdot s) = 2 \cdot s^2 + f + 2 \cdot f \cdot s$$

$$\Rightarrow E(Y) \cdot \left(1 - (1-s) - (1-s) \cdot s\right) = 2 \cdot s^2 + (1-s) + 2 \cdot (1-s) \cdot s \Rightarrow E(Y) \cdot s^2 = s + 1 .$$

So, $E(Y) = \frac{1}{s} + \frac{1}{s^2} = 6 + 36 = 42$.

e) Let p_A be the probability that Anna wins the game, i.e., event A happens before event B when throwing two dice repeatedly. We observe that events A and B cannot happen simultaneously, and neither can A and S (sum 7).

If, on the first roll, event A happens, then the probability that A happened before B is 1 (it is a sure event). This happens with a probability of $p=\frac{1}{36}$.

If we have sum 7 on the first roll and then event A happens on the second turn, then the probability of A happening before B is 1, because A happened and B did not. This case has the probability $s\cdot p=\frac{1}{6}\cdot\frac{1}{36}$ of happening.

Else, if we get sum 7 on both the first two rolls, i.e., B happens in the first two rolls, event whose probability is $s\cdot s=q=\frac{1}{36}$ then the probability of A happening before B would be 0.

In the case (with probability $s\cdot(1-p-s)=\frac{1}{6}\cdot\frac{29}{36}$) that we get sum 7 on the first turn, and then neither sum 7 nor A happen on the second turn, the probability of A happening before B is p_A, since the first two turns did not affect the game (we didn't make any steps toward B or A happening).

Also, in the case (with probability $1-p-s=\frac{29}{36}$) that neither sum 7 nor A happen on the first turn, the game was not affected by the first turn. That means that the probability that A happens before B in the rolls that follow is p_A.

All of these cases, combined, form a complete system of events.

Therefore, we have that

$$p_A=\frac{1}{36}\cdot 1+\frac{1}{6}\cdot\frac{1}{36}\cdot 1+\frac{1}{36}\cdot 0+\frac{1}{6}\cdot\frac{29}{36}\cdot p_A+\frac{29}{36}\cdot p_A.$$

Thus, $p_A=\frac{7}{13}$, which means that Anna's odds of winning are $7:6$.

f) Although both events A and B have the same chance of occurring (that is, $\frac{1}{36}$), when conditioning them, one conditioned by the other one, the event A has more chances to occur before event B has occurred.

However, this result is expected, as event B has a chance of occurring if at least two rolls of dice are made, while event A can occur even in one roll.

g) **Method I**

Knowing that the first roll of the dice was not the double $(6,6)$, the results we can get are from the set $\left\{(x,y)\,\middle|\, x,y=\overline{1,6}\right\}\setminus\{(6,6)\}$, which has 35 elements. The results from this set that have sum 7 are

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

We get that the probability that on the first roll we get sum 7 is $s' = \frac{6}{35}$. The probability that we do not is $f' = 1 - s' = \frac{29}{35}$.

If we did not get sum 7 on the first turn, the game was not affected by the first turn. That means that the probability that *A* happens before *B* is still $p_A = \frac{7}{13}$, as we have proved at point e).

Else, if we get sum 7 on the first turn, we have three cases: either we get sum 7 on the second turn, or we get $(6,6)$, or we do not get any of these. In the first case (with probability $s = \frac{1}{6}$), *B* would happen in the second turn, and *A* would not have happened by then. So, the probability that event *A* happens before event *B* does would be 0. In the second case (with probability $s = \frac{1}{36}$, *A* would happen before *B*, thus Anna wins the game with a probability of 1 (a sure event). In the third case (with probability $1 - p - s = \frac{29}{36}$), the game would not be affected by the first two turns, because we did not make any steps toward *A* or *B* happening. Thus, the probability of *A* happening before *B* is $p_A = \frac{7}{13}$, from point e).

All of these cases, combined, form a complete system of events.

Therefore we have that the probability that event *A* happens before event *B*, knowing that the first roll of the dice was not the double $(6,6)$ is

$$p_A{}' = \frac{29}{35}\cdot\frac{7}{13} + \frac{6}{35}\cdot\frac{1}{6}\cdot 0 + \frac{6}{35}\cdot\frac{1}{36}\cdot 1 + \frac{6}{35}\cdot\frac{29}{36}\cdot\frac{7}{13} = \frac{239}{455}.$$

Thus, Anna's odds of winning the game if we know the first roll of the dice was not the double $(6,6)$ are $239:216$.

Method II

Let *M* be the event that Anna wins the game, and let *N* be the event that the first roll is not the double $(6,6)$.

Then,

$$M = M \cap \left(N \cup \bar{N}\right) = \left(M \cap N\right) \cup \left(M \cap \bar{N}\right)$$

and so, as $\left(M \cap N\right) \cap \left(M \cap \bar{N}\right) = \varnothing$, we can write:

$$p(M) = p(M \cap N) + p\left(M \cap \bar{N}\right).$$

Denote by $M|N$ the event that Anna wins the game knowing that the first roll was not $(6,6)$. Then, $M|\bar{N}$ the event that Anna wins the game knowing that the first roll was $(6,6)$.

By the definition of a conditional probability, $p\left(M|N\right)=\dfrac{p\left(M\cap N\right)}{p\left(N\right)}:=p'_A$, the required probability here. Thus,

$$p\left(M\right)=p\left(M|N\right)\cdot p\left(N\right)+p\left(M|\bar{N}\right)\cdot p\left(\bar{N}\right)$$

$$\Rightarrow p_A=p'_A\cdot(1-p)+1\cdot p$$

$$\Rightarrow \frac{7}{13}=p'_A\cdot\frac{35}{36}+1\cdot\frac{1}{36}\Rightarrow p'_A=\frac{239}{455}.$$

Conclusions

We have answered all the points of the proposed problem. Some of the tasks where solved using more than one method. By solving this problem, we have enlarged our knowledge on probability and random variables. Some of the requirements were very challenging and they required advanced skills in probability. The problem gave us a short insight into the very popular world of betting.

References

1. Stoleriu I. *Probabilități și statistică matematică. Note de curs.* Universitatea „Al. I. Cuza” Iași, 2016