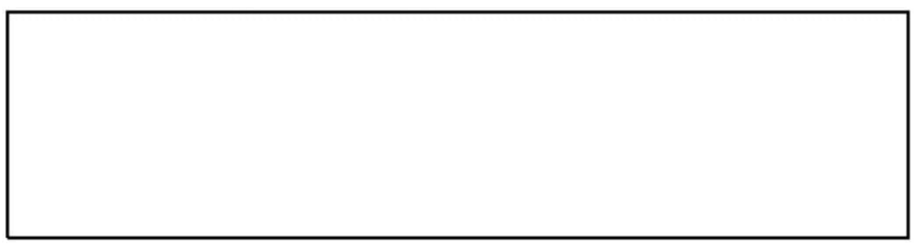
Equidecomposability of polygons

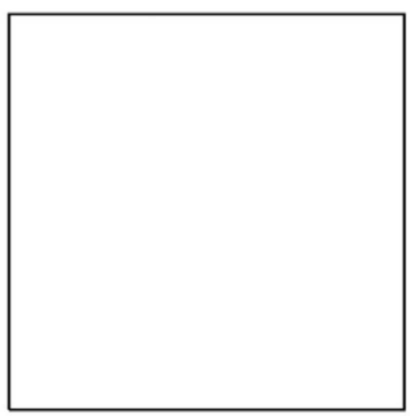
Statement: Two polygons A and B of equal area are given. Can A be cut into smaller polygons so that after rearrangement they form B? To make the redaction easier, we will use the following notations: $A^{\star}B$, which means that A *can* be cut into polygons which rearranged form B, and $A!^{\star}B$, which means that A *cannot* be cut into polygons which rearranged form B.

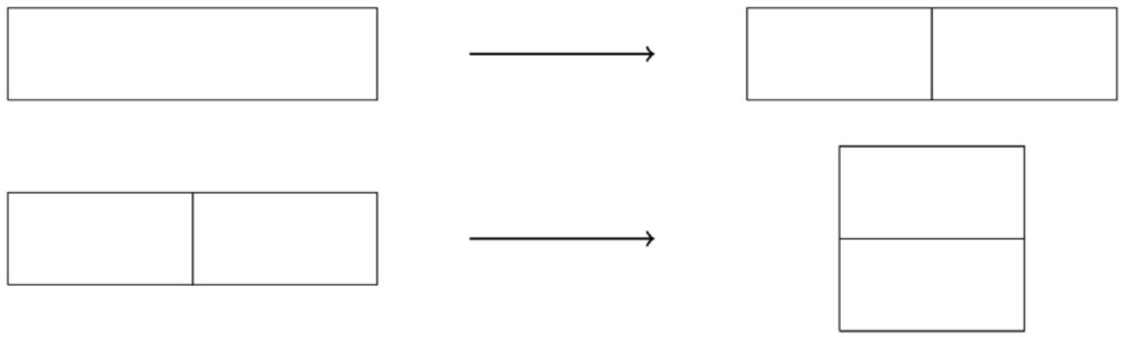
Example 1:

A= rectangle with sides 1 and 4



A’= square with side 2





A⋆B

Example 2:

A = square with side 2

B = square with side 3

A!⋆B

Remarks :

- If A⋆B, then B⋆A;
- If A⋆C and C⋆B, then A⋆B;

From these two remarks → if A⋆C and B⋆C, then A⋆B.

The actions of cutting and rearranging do not affect the area of the initial polygon. In order for A⋆B, it is absolutely necessary that the area of A is equal to the area of B.

We will prove that given two polygons A and B of equal areas, there is a way to cut A into smaller polygons that rearranged form B.

Steps:

I. *A can be cut into triangles;*
II. *Any given triangle can be cut into smaller polygons that rearranged form a rectangle with equal area;*
III. *Any given rectangle can be cut into smaller polygons which rearranged form a rectangle with one side of length 1 and of equal area;*
IV. *All these rectangles with one side of length 1 can be put together to form a bigger rectangle with one side of length 1 and with the same area as A;*

According to all these steps, we can say that A*C, for any polygon A. B is also a polygon → B can also be cut into smaller polygons which rearranged form C' (rectangle with one side of length 1 and the same area as B). Because A and B have the same area, C and C' are the same polygon. Then: A*C and B*C → A*B.

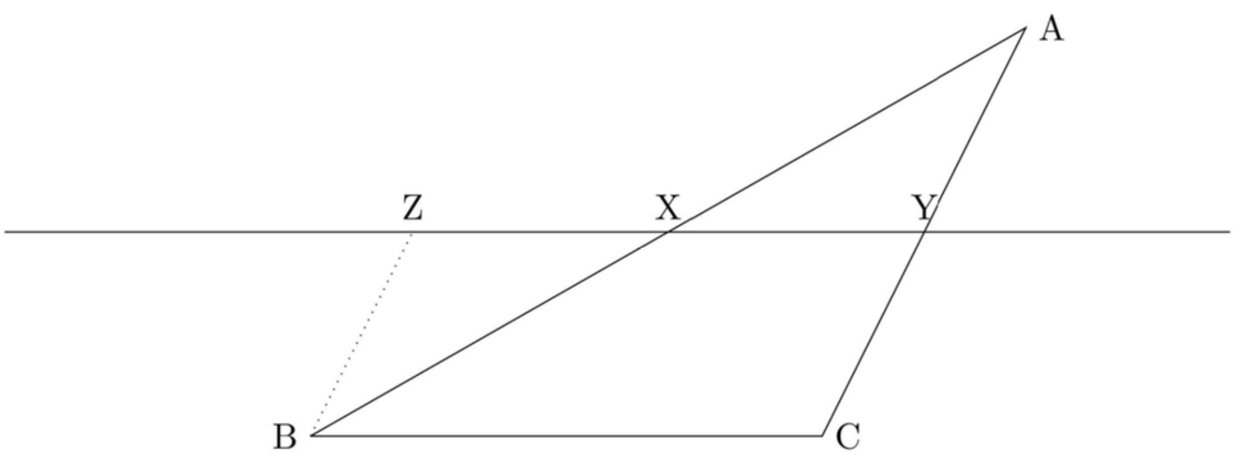
I. The triangulation of a polygon

For a convex polygon, it can be cut into triangles by choosing a vertex and tracing all of the diagonals which start from that vertex. For a concave polygon, it can be cut into convex polygons, and we repeat the previous steps.

II. Transforming any given triangle in a rectangle

a) We will transform the triangle ΔABC in the rectangle ZYCB. We take X and Y as the midpoints of the line segments [AB], respectively [AC]. We take *d* as the parallel to AC, that goes through B, and Z is the intersection of *d* and XY ({Z} = d U XY). XY is the midsegment in ΔABC, XY ∥ BC. XY ∥ BC, ZB ∥ YC →ZYCB is a parallelogram. Because we know that XYCB is ZYCB intersected with ΔABC (XYCB = ZYCB ∩ ΔABC), in order to show that ΔABC*ZYCB, it suffices to prove that ΔZXB ≡ ΔYXA.

X is the midpoint of the line segment [AB] (X = mid[AB]) → AX ≡ BX (1), ∠ZBX and ∠YXA are opposite angles → ∠ZBX ≡ ∠YXA (2); ZB ∥ AY, AB = transversal → ∠ZBX ≡ ∠YAX (alternate interior angles) (3). From (1), (2) and (3) ΔZXB ≡ ΔYXA (angle, side, angle - ASA)



b) We will transform the parallelogram ZYCB into a rectangle ZZ'YY'. Z' is the foot of the perpendicular from Z to BC, and Y' is the foot of the perpendicular from Y to BC.

ZZ'⊥BC; BC ∥ ZY → ZZ' ⊥ ZY → ZZ'=dist(BC, ZY) - (ZZ' is the distance from BC to ZY)

In the same way we prove that YY' is also the distance from BC to ZY. (YY'=dist(BC, ZY)) → ZZ' = YY' (1)
α = m(∠BZY)

$\beta = m(\angle BZZ')$
$\gamma = m(\angle CYY')$
$\theta = m(\angle ZYC)$

$BZ \parallel YC$, ZY = transversal → $\alpha + \theta = 180^\circ$ (i)
$\beta = \alpha - 90^\circ$ (ii)
$\theta + \gamma = 90^\circ$ (iii)
from (i), (ii) and (iii) → $\alpha - \gamma = 90^\circ$ → $\gamma = \alpha - 90^\circ$ → $\gamma = \beta$,
$\angle BZZ' = \angle CYY'$ (2)
$BZ = CY$ (3)
From (1), (2) and (3) → $\Delta BZZ' \equiv \Delta CYY'$ (SAS – side, angle, side)

III. The transformation of a rectangle into a rectangle with one side of length 1

(1) We will prove that any rectangle X' can be transformed into a square with the same area → (2) any square can be transformed into any rectangle of the same area

From (1) and (2) → any rectangle can be transformed into any other rectangle of the same area → any rectangle can be transformed into a rectangle of the same area and with one side of length 1.

We consider the rectangle A’B’C’D’, with sides of length a’ and b’ (a’<b’) and the square of equal area XYZT, with side of length c=$\sqrt{a' \cdot b'}$.

We will transform the rectangle A’B’C’D’ into the rectangle ABCD with sides of lengths a and b, and with the following properties:

a < b;

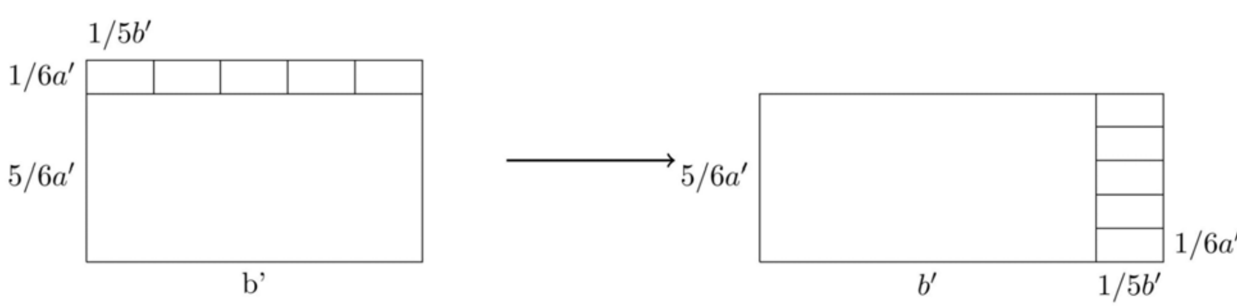
$\frac{b}{c} \in (\sqrt{5}, \sqrt{10})$

(we will explain later on why this condition is necessary)

I. $\frac{b'}{c} \leq \sqrt{5}$

We will do this transformation α times, with α ∈ ℕ.

We cut the initial rectangle into another rectangle of sides a’$\cdot\frac{5}{6}$, respectively b’ and 5 rectangles of sides a’$\cdot\frac{1}{6}$ and b’ $\frac{1}{5}$. These rectangles are then rearranged to form a rectangle with sides of length a’$\cdot\frac{5}{6}$ and b’$\cdot\frac{6}{5}$.



After the α transformations, b will be equal to $b'\cdot\left(\frac{6}{5}\right)^{\alpha}$.

We will prove that $\exists\ \alpha \in \mathbb{N}$, so that $\frac{b'}{c}\cdot\left(\frac{5}{6}\right)^{\alpha} \in (\sqrt{5}, \sqrt{10})$.

Suppose that there isn't any $\alpha \in \mathbb{N}$ that respects the property above

$\rightarrow \exists\ \alpha' \in \mathbb{N}$ so that $\frac{b'}{c}\cdot\left(\frac{5}{6}\right)^{\alpha'} < \sqrt{10}$ and $\frac{b'}{c}\cdot\left(\frac{5}{6}\right)^{\alpha'+1} > \sqrt{5}$.

Let x be $\frac{b'}{c}\cdot\left(\frac{5}{6}\right)^{\alpha'}$.

$$x \geq \sqrt{10} \rightarrow x\cdot\frac{5}{6} \geq \sqrt{10}\cdot\left(\frac{5}{6}\right) (1)$$

$$x\cdot\frac{5}{6} \leq \sqrt{5}\ (2)$$

$$\sqrt{10}\cdot\left(\frac{5}{6}\right) \geq \sqrt{5}\ (3)$$

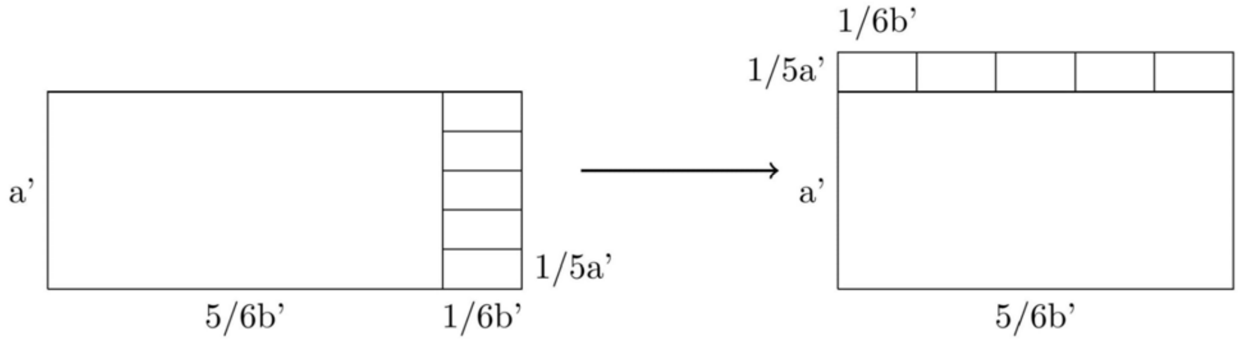
From (1), (2) and (3) → contradiction, so $\exists\ \alpha \in \mathbb{N}$ so that

$\frac{b'}{c}\cdot\left(\frac{5}{6}\right)^{\alpha} \in (\sqrt{5},\sqrt{10})$

II. $\frac{b'}{c} > \sqrt{10}$

We will do the following transformation α times, with $\alpha \in \mathbb{N}$

We will cut the initial rectangle into another rectangle of sides a', respectively $\left(\frac{5}{6}\right)\cdot b'$, and five rectangles of sides $\left(\frac{1}{5}\right)\cdot a'$, respectively $\left(\frac{1}{6}\right)\cdot b'$. These rectangles are rearranged in order to form a rectangle with sides of lengths $\left(\frac{6}{5}\right)\cdot a'$, respectively $\left(\frac{5}{6}\right)\cdot b'$.



After the α transformations, b will be equal to $b' \cdot (\frac{5}{6})^{\alpha}$.

We will prove that ∃ α ϵ ℕ so that $(\frac{b\prime}{c}) \cdot (\frac{5}{6})^{\alpha} \in (\sqrt{5}, \sqrt{10})$

Suppose that there is no α with the requested property

→ ∃ α' ϵ ℕ so that $(\frac{b\prime}{c}) * (\frac{5}{6})^{\alpha'+1} \leq \sqrt{5}$, and $(\frac{b\prime}{c}) \cdot (\frac{5}{6})^{\alpha\prime} \geq \sqrt{10}$.

Let x be $(\frac{b\prime}{c}) \cdot (\frac{5}{6})^{\alpha\prime}$.

$x \geq \sqrt{10} \rightarrow x \cdot (\frac{5}{6}) \geq \sqrt{10} \cdot (\frac{5}{6})$ (1)

$x \cdot (\frac{5}{6}) \leq \sqrt{5}$ (2)

$\sqrt{10} \cdot (\frac{5}{6}) \geq \sqrt{5}$ (3)

From (1), (2) and (3) → contradiction. Therefore, ∃ α ϵ ℕ so that

$(\frac{b\prime}{c}) \cdot (\frac{5}{6})^{\alpha} \in (\sqrt{5}, \sqrt{10})$.

Let ABCD be a rectangle with sides of lengths a and b, respectively XYZT a square with side of length c = $\sqrt{a \cdot b}$ with the following properties:

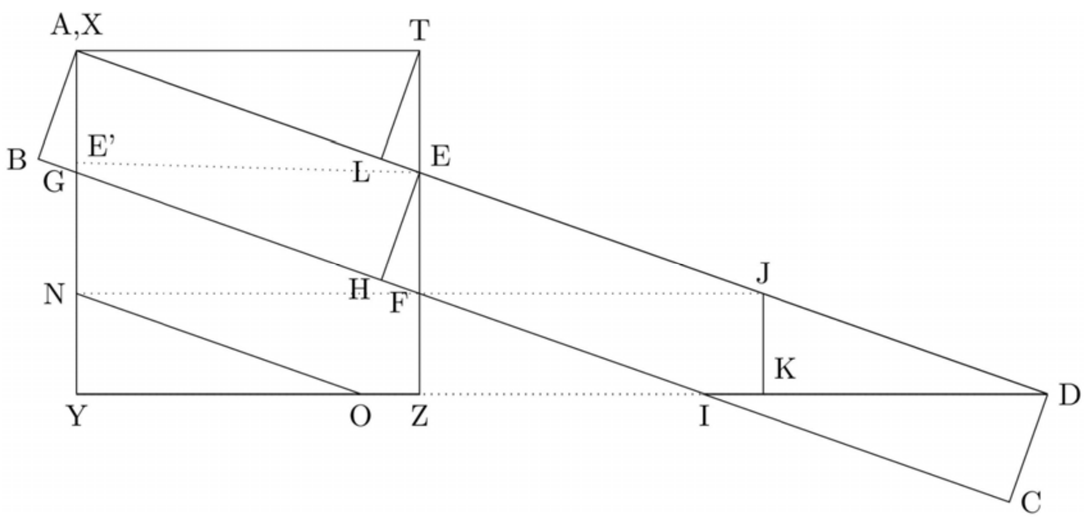
$$a < b;$$

$$\left(\frac{b}{c}\right) \in (\sqrt{5}, \sqrt{10})$$

$$A_{ABCD} = A_{XYZT}$$

A’B’C’D’⋆ABCD

If we prove that ABCD⋆XYZT → A’B’C’D’ ⋆ XYZT



We overlap ABCD and XYZT so that D ∈ (YZ) and A is the same as X.

In order to prove that ABCD⋆XYZT, we will prove that there is a way to cut the rectangle ABCD so that the resulting polygons can form XYZT after rearrangement.

For this we will cut both XYZT and ABCD and we will prove that each polygon resulted from the dissection of the square is congruent with a polygon resulted from the dissection of the rectangle.

We will use the following notations:

$y = m(\angle YDA)$

F = BC ∩ TZ

H = the foot of the perpendicular from E to BC

N = symmetric to A with respect to G

K = the foot of the perpendicular from J to YD

L = the foot of the perpendicular from T to AD

l = the length of the line segment AE

E = AD ∩ TZ

G = BC ∩ AY

I = BC ∩ YD

J = symmetric to A with respect to E

o = the parallel to BC through N

O = o ∩ YZ

d = the length of the line segment AG

XYZT is cut into {AEFG; TLE; GFZON; NOY; TLA}

ABCD is cut into {AEFG; ABG; EJKIF; JDK; DCI}

We will prove that:

(1)TLE ≡ ABG
(2)GFZON ≡ EJKIF
(3)NOY ≡ IDK
(4)TLA ≡ DCI
(5)AEFG ≡ AEFG

From (1), (2), (3), (4) and (5) results that there is a way to cut ABCD into polygons that rearranged form XYZT

AY ⊥ YD → Δ AYD is a right triangle → sin(y) $= \frac{AY}{AD}$ = $\frac{c}{b}$

EZ ⊥ ZD → ΔEZD right triangle → m($\angle ZED$)=90˚- y
EH ⊥ BC, AD ∥ BC → EH ⊥ AD → m($\angle HED$)=90˚ (1)
m(∠HED) = m(∠HEF) + m(∠ZED) (2)
m(∠HEF) = y (3)
EH ⊥ HF → ΔEHF is a right triangle (4)
From (1), (2), (3) and (4) → cos(y)= $\frac{EH}{EF}$ (i)
EH ⊥ BC, BC ∥ AD → EH = distance from BC to AD (1)
AB ⊥ BC, BC ∥ AD → AB = distance from BC to AD (2)
From (1) and (2) → AB = EH = a (ii)
EF ∥ AG, AE ∥ GF → AEFG is a parallelogram → EF = AG = d (iii)
From (i), (ii) and (iii) → cos(y)=$\frac{a}{d}$
E' = the foot of the perpendicular from E to AY
EE' ⊥ AY, ZT ∥ AY → EE' = the distance from AY to ZT → EE'=c
ZT ∥ AY, YZ ⊥ AY → YZ = the distance from AY to ZT → YZ=c
YZ = EE' = c (1)
ΔAE'E = right triangle (2)
In ΔAYD:
m(∠AYD) = 90˚, m(∠ADY) = y → m(∠YAD) = 90˚ - y = m(∠E'AE) (3)
From (2) and (3) → m(∠ AEE') = y (4)
From (1), (2) and (4) → cos(y) = $\frac{GE}{AE}$ = $\frac{c}{l}$, sin(y) = $\frac{AE'}{l}$
A_{ABCD}= a·b, A_{XYZT}= c^2 → ab = $\frac{c^2}{b \cdot c}$→ $\frac{a}{c}$ = $\frac{c}{b}$ = sin(y)

sin(y)·[cos(y)/cos(y)]= $\left(\frac{a}{c}\right) \cdot \left(\frac{c}{l}\right) \cdot \left(\frac{d}{a}\right) = \frac{d}{l}$ → sin(y) = $\frac{d}{l}$, sin(y)=$\frac{AE'}{l}$

→ AE' = d, and G, E' ϵ [AY] → E' = G

EG ⊥ AY → EG ∥ YD

AEFG is a parallelogram → EF = AG, AG = GN → EF = GN

AE ∥ GF, AG transversal → m(∠GAE) = m(∠NGF) = 90˚ - y, but m(∠FEJ) = 90˚ - y → ∠ NGF ≡ ∠ FEJ

EJ ∥ FI, EF transversal → m(∠FEJ) + m(∠EFI) = 180˚ → m(∠EFI) = 90˚+ y

GF ∥ NO, GN transversal → m(∠NGF) + m(∠GNO)=180˚

→ m(∠GNO) =90˚+y

∠EFI ≡ ∠GNO

∠EFI ≡ ∠GFZ (opposite angles)

ΔJKD is a right triangle, m(∠KDJ) = y → m(∠KJD) = 90˚ - y

∠KJE and ∠KJD are supplementary angles

→ m(∠KJE) = 90˚ +y = m(∠EFI)

∠KJE ≡ ∠EFI, ∠EFI ≡ ∠GFZ → ∠KJE ≡ ∠GFZ

JK ⊥ YD, I ϵ YD → m(∠JKI) = 90˚

FZ ⊥ YD, O ϵ YD → m(∠FZO) = 90˚

NY ⊥ YZ, FZ ⊥ YZ → NY ∥ FZ

YZ ≡ ZI (they're actually the same line)

NY ∥ FZ, NO ∥ FI, YO ≡ ZI → ΔNYO ~ ΔFZI → ∠NOY ≡ ∠FIZ

∠NOZ and ∠NOY supplementary, ∠FIK and ∠FIZ supplementary, ∠NOY ≡ ∠FIZ → FIK ≡ NOZ

∠NGF ≡ ∠FEJ, ∠GNO ≡ ∠EFI, ∠GFZ ≡ ∠EJK, ∠FZO ≡ ∠JKI, ∠ZON ≡ ∠KIF

→ EJKIF ~ GFZON, EF = GN → EJKIF ≡ GFZON → JK = FZ, FI = NO, IK=OZ

NO ∥ BC, BC ∥ AD → NO ∥ AD

NY ⊥ YD, JK ⊥ YD → NY ∥ JK

NO ∥ AD, NY ∥ JK, YO ≡ KD → ΔNYO~ ΔJKD, ΔNYO ~ΔFZI

→ ΔJKD ~ ΔFZI, JK = FZ → ΔJKD ≡ ΔFZI

ΔNYO ~ Δ FZI, FI = NO → ΔNYO ≡ ΔFZI, ΔJKD ≡ ΔFZI → ΔNYO ≡ ΔJKD

→ YO = KD

C =YO + OZ = KD + IK = ID

m(GAD) = 90˚ - y, ∠GAD and ∠DAT complementary angles

→ m(∠DAT) = y

ΔFZI ≡ ΔJKD → m(∠FIZ) = m(∠JDK) = y, m(∠CID) = m(∠FIZ) (opposite angles) → m(∠CID) = y = m(∠LAT)

ΔLAT and ΔCID are right triangles AT = ID, ∠LAT ≡ ∠CID

→ ΔLAT ≡ ΔCID → LT = CD

ABCD = rectangle → AB = CD = LT

∠BGA and ∠AGF are supplementary angles, ∠LET and ∠LEF are supplementary angles, m(∠AGF) = m(∠LEF) = 90˚ + y → ∠BGA ≡ ∠LET

ΔABG and ΔTLE = right triangles, AB = LT → ΔABG ≡ Δ TLE

AEFG ≡ AEFG, ABG ≡ TLE, EJKIF ≡ GFZON, IDK ≡ NOY, DCI ≡ TLA

→ ABCD*XYZT → A'B'C'D'*XYZT and XYZT*A'B'C'D'.

Therefore, any rectangle can be transformed into a square of equal area, and a square can be transformed into any rectangle of equal area. In other words, any rectangle can be transformed into another rectangle of equal area.

In conclusion, this is the procedure (not necessarily the best one) to dissect a polygon A into smaller polygons which rearranged form B.

Recap:

1) We start by cutting A in triangles (we proved at I that it's possible for any given polygon)
2) We transform each triangle firstly in a parallelogram of equal area, and then in a rectangle of the same area. (we proved at II that it's possible for any triangle)

3) We transform each rectangle in a square of equal area, which will be then transformed into a rectangle with one side of length 1 (or any other length as long as it's the same throughout the whole process) (we proved at III that it's possible for any square). We put together all the rectangles with one side of length 1 to form a bigger rectangle (a) with one side of length 1 and the same area as A.
4) We repeat steps 1), 2) and 3) for polygon B, to obtain a rectangle with one side of length 1 and the same are as B (b). Because B and A have the same area, rectangle a and rectangle b will be the same rectangle. In order to obtain B from A, we repeat the steps in reverse order.

Now, here's why we need $\frac{b}{c} \in (\sqrt{5}, \sqrt{10})$ in order to use this demonstration:

In order to consider the pentagon EJKIF, I needs to be more to the left of K (YI < YK), thus the line segments [JK], respectively [IF] intersect.

In ΔGIY we observe that YI = ctg(y)·GY = ctg(y)·(c-d), NY = JK, NY ∥ JK

→ NYKJ is a parallelogram → YK = NJ, YK ∥ NJ, YK ⊥ AY → NJ ⊥ AY

→ ΔANJ is a right triangle

m(NAJ) = 90˚ - y, m(JNA) = 90˚ → m(NJA) = y

→ NJ = cos(y)·AJ = cos(y)*2l, YK = cos(y)·2l

cos(y)·2l > $\frac{\cos(y)}{\sin(y)} \frac{c-d}{\sin(y)}$

cos(y)·2l·($\frac{d}{l}$) > cos(y)·(c-d)

2d > c-d → 3d > c → 3 > $\frac{c}{d}$, d = $\frac{a}{\cos(y)}$ → 3 > c·$\frac{\cos(y)}{a}$

$\frac{c}{a}$ = $\frac{b}{c}$= $\frac{1}{\sin(y)}$ → 3 > $\frac{\cos(y)}{\sin(y)}$→ 3 > ctg(y)

In order to consider the pentagon GNOZF, F needs to be higher than Z (F ϵ [TZ]). We will consider x to be the distance (with sign) from F to Z (positive if F is higher than Z, and negative if F is lower than Z).

TE + EF +FZ = TZ

TE = EF = d, TZ = c

2d + FZ = c, and we want FZ > 0 → 2d < c → 2 < $\frac{c}{d}$ and through the same steps → 2 < ctg(y)

ctg(y) ϵ (2, 3), y ϵ (0˚, 90˚), ctg is a monotonically decreasing function on (0˚, 90˚) → y ϵ (arcctg(3), arcctg(2))

sin is a monotically increasing function on (0˚, 90˚)

→ sin(y) ϵ (sin(arcctg(3)), sin(arcctg(2))), therefore sin(y) ϵ ($\frac{1}{\sqrt{10}}$, $\frac{1}{\sqrt{5}}$)

sin(y) = $\frac{c}{b}$ → $\frac{b}{c}$ ϵ ($\sqrt{5}$, $\sqrt{10}$)

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