Bots on Grid

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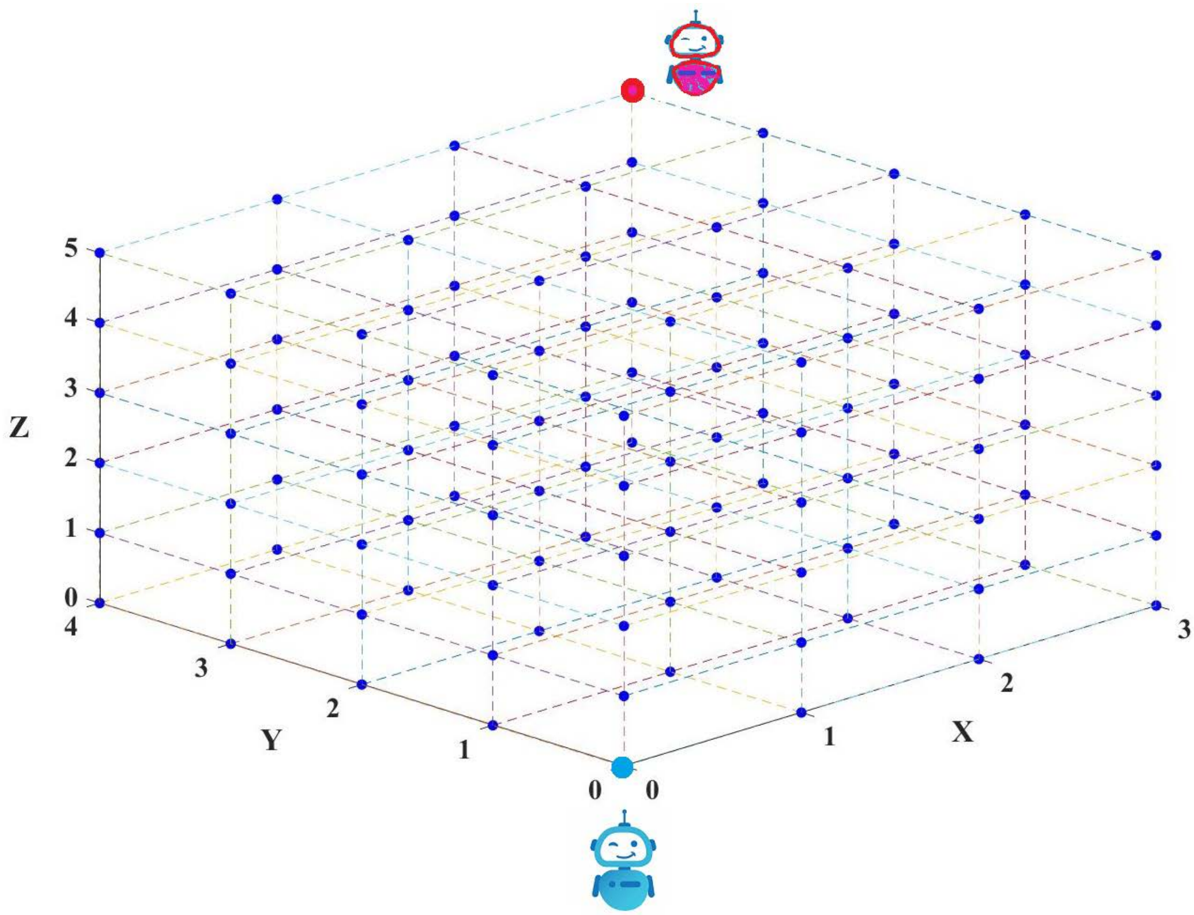
1. PRESENTATION OF THE RESEARCH TOPIC

Given a (x, y, z) three-dimensional grid and two different bots moving one unit per step, towards the opposite corner, at the same pace, starting at the same time, from points $(0,0,0)$ and (x, y, z), we try to find the chances of the two bots meeting on one of the grid's points.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

In a video game, two bots are walking on the three-dimensional integer grid plotted below, where the lower corner is at $(0,0,0)$. One bot is programmed to start from

$(0,0,0)$, the other bot starts from the opposite corner, $(3,4,5)$. In one step, each bot can walk only one unit, at the same pace, in either the *X*, *Y* or *Z* direction. All possible trajectories from (0,0,0) to (3,4,5) are considered to have equal probabilities. The first bot can only increase its coordinate (in one of the three directions) but never decreases it, while the second bot can only decrease its coordinate (in one of the three directions) but never increases it. What are the chances that the two bots meet half-way, on the same node of the grid? What happens if the movement's direction of the robots is chosen with equal probability at each step? Generalize the problem if possible.



3. THE SOLUTION

Let $0 \le x \le 3, 0 \le y \le 4, 0 \le z \le 5$.

- Suppose that, at each step, Blue can only move 1 unit upwards in either the *x*-, the *y*- or the *z*-direction. Then, the number of paths that Blue can follow through the 3-D lattice when travelling from the origin $O(0,0,0)$ to the point $P(x,y,z)$ is:

$$B_{(x,y,z)} = \frac{(x+y+z)!}{x!y!z!}.$$

- Suppose that, at each step, Red can only move 1 unit downwards in either the *x*-, the *y*- or the *z*-direction. Then, the number of paths of Red can follow through the 3-D lattice when travelling from the point $M(3,4,5)$ to the point $P(x,y,z)$ is:

$$R_{(x,y,z)} = \frac{(3-x+4-y+5-z)!}{(3-x)!(4-y)!(5-z)!}.$$

(just imagine that the point *M* is the new the origin *O'*)

Clearly, the two bots will meet after $\frac{3+4+5}{2}=6$ steps. We shall call a *favorable path* a situation when Blue and Red meet at a point $P(x,y,z)$ situated on any level z $(0 \le z \le 5)$. By the product rule, as the two bots travel independently on the grid, the number of favorable paths associated to a particular point $P(x,y,z)$ is equal to the product between the number of paths that each bot can follow to the meeting point. There are 18 possible meeting points on the grid, and we count all the trajectories that each bot can follow until they meet (see last column in Table 1). The total number of favorable paths, *F,* is equal to the sum of all favorable paths for each of the 18 points. From Table 1, we see that $F = 27720$.

The number of all equiprobable possible trajectories that the two bots can take is the product of the total number of possible paths that each bot can take after 6 steps, regardless they meet or not. From Table 1, we see that:

$$T = 642 \cdot 642 = 412164.$$

Then, the probability that the two bots meet halfway of the grid is equal to

$$P = \frac{F}{T} = \frac{27720}{412164} = 0.0673.$$

that is, there are **6.73%** chances to meet.

Level for *z* direction	Possible meeting nodes (x,y,z)	Number of trajectories for Blue after 6 steps	Number of trajectories for Red after 6 steps	Total number of meetings for the level
z = 5	$(0,1,5)$	$\frac{6!}{0!1!5!}=6$	$\frac{6!}{3!3!0!}=20$	$6\cdot 20=120$
	$(1,0,5)$	$\frac{6!}{1!0!5!}=6$	$\frac{6!}{2!4!0!}=15$	$6\cdot 15=90$

$z = 4$	$(0,2,4)$	$\frac{6!}{0!2!4!}=15$	$\frac{6!}{3!2!1!}=60$	$15\cdot60=900$
	$(1,1,4)$	$\frac{6!}{1!1!4!}=30$	$\frac{6!}{2!3!1!}=60$	$30\cdot60=1800$
	$(2,0,4)$	$\frac{6!}{2!0!4!}=15$	$\frac{6!}{1!4!1!}=30$	$15\cdot30=450$
$z = 3$	$(0,3,3)$	$\frac{6!}{0!3!3!}=20$	$\frac{6!}{3!1!2!}=60$	$20\cdot60=1200$
	$(1,2,3)$	$\frac{6!}{1!2!3!}=60$	$\frac{6!}{2!2!2!}=90$	$60\cdot90=5400$
	$(2,1,3)$	$\frac{6!}{2!1!3!}=60$	$\frac{6!}{1!3!2!}=60$	$60\cdot60=3600$
	$(3,0,3)$	$\frac{6!}{3!0!3!}=20$	$\frac{6!}{0!4!2!}=15$	$20\cdot15=300$
$z = 2$	$(0,4,2)$	$\frac{6!}{0!4!2!}=15$	$\frac{6!}{3!0!3!}=20$	$15\cdot20=300$
	$(1,3,2)$	$\frac{6!}{1!3!2!}=60$	$\frac{6!}{2!1!3!}=60$	$60\cdot60=3600$
	$(2,2,2)$	$\frac{6!}{2!2!2!}=90$	$\frac{6!}{1!2!3!}=60$	$90\cdot60=5400$
	$(3,1,2)$	$\frac{6!}{3!1!2!}=60$	$\frac{6!}{0!3!3!}=20$	$60\cdot20=1200$
$z = 1$	$(1,4,1)$	$\frac{6!}{1!4!1!}=30$	$\frac{6!}{2!0!4!}=15$	$30\cdot15=450$
	$(2,3,1)$	$\frac{6!}{2!3!1!}=60$	$\frac{6!}{1!1!4!}=30$	$60\cdot30=1800$
	$(3,2,1)$	$\frac{6!}{3!2!1!}=60$	$\frac{6!}{0!2!4!}=15$	$60\cdot15=900$
$z = 0$	$(2,4,0)$	$\frac{6!}{2!4!0!}=15$	$\frac{6!}{1!0!5!}=6$	$15\cdot6=90$
	$(3,3,0)$	$\frac{6!}{3!3!0!}=20$	$\frac{6!}{0!1!5!}=6$	$20\cdot6=120$
Total number of paths		642	642	**27720**

Table 1: Number of paths the two bots can follow on the grid after 6 steps

After that, we started to think what happens if the movement's direction of the robots is chosen with equal probability at each step. In this case, the trajectories will not have the same probability anymore.

In order to familiarize with the new problem, we study a few particular cases:

Case I. Two lengths are zero.

In this case, its dimensions can be either $(x,0,0)$, $(0,x,0)$ or $(0,0,x)$. We will only consider the case $(x,0,0)$ because the other two can be solved similarly.

The parallelepiped is now a segment of length x.

i) $2\,|\,x \Rightarrow x = 2k+1, k \in \mathbb{N}^*$

In this case, assuming that the two robots move at a speed of 1 unit/second, the two of them will meet after k seconds, at point $(k,0,0)$, no matter what their route is.

Therefore, the probability is 1.

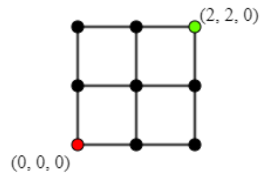
ii) $2 \nmid x \Rightarrow x = 2k+1, k \in \mathbb{N}$

Every time, they will meet at the middle point of the segment. Because this point is at a non-integer distance from each of the two starting points, it is not one of the nodes.

Therefore, the probability equals 0 in this situation as they never meet at one of the nodes.

Case II. One length is zero.

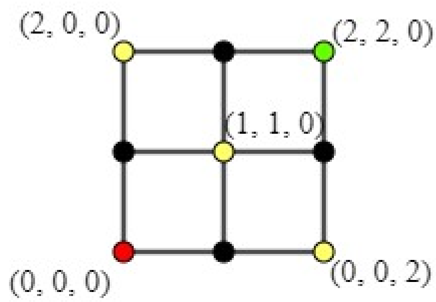
We will analyze a particular case which is easier to understand: a 2×2 square.



One of the robots starts at point $(0,0,0)$ (the red point) and the other one starts at point $(2,2,0)$ (the green point). For this case, we will call them Red, respectively Green.

We will now find the number of favorable cases, c_f. We assume that they meet.

The two of them will meet, assuming they move at a speed of 1 segment/ second, after $\frac{2+2}{2}=2$ seconds. Therefore, they can only meet at point $(0,2,0)$, $(1,1,0)$ or $(2,0,0)$. These are the yellow points in the following figure.



Situation 1. They meet at point $(0,2,0)$. Each of them can get to this point in 1 way, therefore in this situation we have $1\cdot 1=1$ favorable case.

Situation 2. They meet at point $(1,1,0)$. Each of them can get to this point in 2 ways, therefore in this situation we have $2\cdot 2=4$ favorable cases.

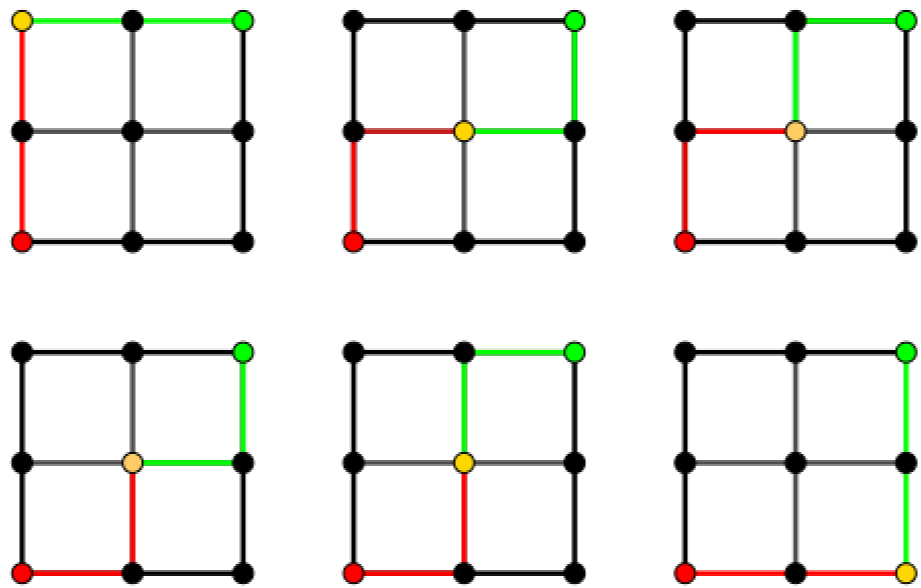
Situation 3. They meet at point $(2,0,0)$. This is symmetrical with Situation 1, therefore there is 1 favorable case as well.

So, $c_f=1+4+1=6$.

Because the number of ways they can move after they meet will be multiplied with both the number of favorable and possible cases and it will simplify when we compute the probability, for the number of possible cases, c_p, we only need to find the number of ways in which they can move $\frac{2+2}{2}=2$ segments. This is, for each of them, 2^2.

Therefore, $c_p = 2^2 \cdot 2^2 = 16$.

Finally, the probability is $p=\frac{c_f}{c_p}=\frac{6}{16}=\frac{3}{8}=37.5\%$.



Case III. The sum of the parallelepiped's dimensions is odd in a general case

We noticed that sometimes the robots never meet. Thus, we want to prove that the sum of the coordinates must be even in order to be possible for the robots to meet. We will prove that if the sum of the coordinates is odd the robots cannot meet.

Let's suppose that the sum of the coordinates (S) is $2k+1$.

At every of the first k steps, the difference between the sums of each of the 2 robots' coordinates shrinks by two (because each of the robots advances by 1). When robot 1 advances with one step, the sum of its coordinates increases by 1, while for robot 2 the sum decreases by 1. After k steps, this will happen:

Robot 1: the sum of his coordinates will be k.

Robot 2: the sum of his coordinates will be $k+1$.

At the next step the values of the sums of their coordinates switch and that means they will never meet. So, if the sum of the coordinates is odd, the probability of the robots meeting is 0.

Case IV. A $3\times4\times5$ parallelepiped.

Notation:

0 = the *X* coordinate is increased by 1 (example: $(0,3,2)$ becomes $(1,3,2)$ = robot moves from $(0,3,2)$ to $(1,3,2)$)

1= the *Y* coordinate is increased by 1 (example: $(2,1,2)$ becomes $(2,2,2)$ = robot moves from $(2,1,2)$ to $(2,2,2)$)

2= the *Z* coordinate is increased by 1 (example: $(0,2,1)$ becomes $(0,2,2)$ = robot moves from $(0,2,1)$ to $(0,2,2)$)

Because the robots can only meet after (3 + 4 + 5) / 2 = 6 steps, we are going to analyze only the first 6 steps: we will have a sequence of 6 numbers (0, 1 or 2) that will show how the first robot (the one that starts at $(0,0,0)$) behaves.

Example: 0 0 1 2 2 1 describes the following series of movements:

$$(0,0,0)\to(1,0,0)\to(2,0,0)\to(2,1,0)\to(2,1,1)\to(2,1,2)\to(2,2,2)$$

We will say that the first robot moves x,y,z if it makes *x* steps on the *Ox* axis, *y* on the *Oy* axis and *z* on the *Oz* axis. We will also say that the second robot moves x,y,z if it decreases with *x* steps on the *Ox* axis, *y* on the *Oy* axis, and *z* on the *Oz* axis.

We will divide the problem for 3,4,5 into multiple cases, depending on how many 0s we have in the sequence of six steps (3, 2, 1 or 0).

For all the sequences, the following rules apply:

1. There cannot be more than three 0s, four 1s or five 2s (when a coordinate is equal to its maximum value, it cannot increase anymore);
2. Every sequence is going to have a probability P;

3. If all the coordinates can increase, at each new step (new number in the sequence) P is going to be multiplied by $\frac{1}{3}$ (all of the 3 the coordinates have an equal probability to increase);
4. If one of the coordinates cannot increase anymore, the probability of the sequence will be multiplied by $\frac{1}{2}$ (both of the remaining coordinates have an equal probability to increase).

In each case, we will write all the sequences that follow the rules and compute their probabilities.

After each case, we will add all of the probabilities of each sequence that ends in the same coordinate.

Observation:

If the first robot is at (x, y, z), the second robot should be at the same coordinates (also (x, y, z)) in order to meet, meaning that it should have moved $3-x, 4-y, 5-z$. So, to get our final probability, we are going multiply the probabilities of the first robot moving x, y, z with the probabilities of the second robot moving $3-x, 4-y, 5-z$ and add all these products together.

Subcase 1: There are three 0s in a sequence:

a. The third 0 appears in the 3rd position: 000 _ _ _

Each of the following sequences has the respective probability $\left(\frac{1}{3}\right)^3 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{216}$ (because for three of the moves, the robot can move in either one of the directions and for the other three it can only move in two directions):

0 0 0 1 1 1, which means the robot is at point (3, 3, 0)
0 0 0 1 1 2
0 0 0 1 2 2
0 0 0 2 2 2.

In the formula of the probability, the $\left(\frac{1}{2}\right)^3$ means that each of the three non-zero digits can be either 1 or 2, because the robot cannot go out of the parallelepiped after these six steps.

b. The third 0 appears in the 4th position: _ _ _0_ _

For each of the following possible digit combinations, the number of possible positions for the 0s is 3 and the probability is $\left(\frac{1}{3}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{324}$.

_ _ _ _ 0 _

1 1 1 $\rightarrow$ 1

1 1 2 $\rightarrow$ 3

1 2 2 $\rightarrow$ 3

2 2 2 $\rightarrow$ 1

c. The third 0 is in the 5th position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 6 and the probability is $\left(\frac{1}{3}\right)^5 \cdot \left(\frac{1}{2}\right)^1 = \frac{1}{486}$:

_ _ _ _ 0 _

1 1 1 $\rightarrow$ 1

1 1 2 $\rightarrow$ 3

1 2 2 $\rightarrow$ 3

2 2 2 $\rightarrow$ 1

d. The third 0 is in the 6th position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 10 and the probability is $\left(\frac{1}{3}\right)^6 \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{729}$:

_ _ _ _ _ 0

1 1 1 $\rightarrow$ 1

1 1 2 $\rightarrow$ 3

1 2 2 $\rightarrow$ 3

2 2 2 $\rightarrow$ 1

So, the final cases and their probabilities are:

$$(3,0,3) \to \frac{1}{216}+\frac{1}{324}+\frac{1}{486}+\frac{1}{729}=\frac{233}{5832}$$

$$(3,1,2) \to \frac{3}{216}+\frac{3}{324}+\frac{3}{486}+\frac{3}{729}=\frac{233}{1944}$$

$$(3,2,1) \to \frac{3}{216}+\frac{3}{324}+\frac{3}{486}+\frac{3}{729}=\frac{233}{1944}$$

$$(3,3,0) \to \frac{1}{216}+\frac{1}{324}+\frac{1}{486}+\frac{1}{729}=\frac{233}{5832}$$

Subcase II. There are two 0s in the sequence.

a. The second 0 appears in the second position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 1 and the associated probability for each of the following sequences is $\left(\frac{1}{3}\right)^6 = 729$:

_ 0 _ _ _ _

1 1 1 1 → 1 (after we choose the positions for the 1s the 2s are fixed. The number of ways in which we can arrange the 1s is $\binom{4}{4}=1$)

1 1 1 2 → 4 (after we choose the positions for the 1s the 2s are fixed, the number of ways in which we can arrange the 1s is $\binom{4}{3}=4$)

1 1 2 2 → 6 (after we choose the positions for the 1s the 2s are fixed, the number of ways in which we can arrange the 1s is $\binom{4}{2}=6$)

1 2 2 2 → 4 (after we choose the positions for the 1s the 2s are fixed, the number of ways in which we can arrange the 1s is $\binom{4}{1}=4$)

2 2 2 2 → 1 (after we choose the positions for the 2s, the number of ways in which we can arrange the 1s is $\binom{4}{0}=1$)

b. The second 0 is in the third position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 2 and the associated probability for each of the following sequences is $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$:

_ _ 0 _ _ _

1 1 1 1 → 1

1 1 1 2 → 4

1 1 2 2 → 6

1 2 2 2 → 4

2 2 2 2 → 1

c. The second 0 is in the fourth position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 3 and the associated probability for each of the following sequences is $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$:

_ _ _ 0 _ _

1 1 1 1 → 1

1 1 1 2 → 4

1 1 2 2 → 6

1 2 2 2 → 4

2 2 2 2 → 1

d. The second zero is in the fifth position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 4 and the associated probability for each of the following sequences is $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$:

_ _ _ _ 0 _

1 1 1 1 → 1

1 1 1 2 → 4

1 1 2 2 → 6

$1\ 2\ 2\ 2 \rightarrow 4$

$2\ 2\ 2\ 2 \rightarrow 1$

e. The second zero is in the sixth position:

For each of the following possible digit combinations, the number of possible positions for the 0s is 5 and the probabilities are

_ _ _ _ _ 0

$$1\ 1\ 1\ 1\ 0 \rightarrow \left(\frac{1}{3}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{324}$$

$$4 \cdot \begin{Bmatrix} 1\,1\,1\,0\,1 \\ \ldots \\ 0\,1\,1\,1\,1 \end{Bmatrix} \rightarrow 4 \cdot \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{2}{243}$$

$$5 \cdot \begin{Bmatrix} 1\,1\,1\,2 \\ 1\,1\,2\,2 \\ 1\,2\,2\,2 \\ 2\,2\,2\,2 \end{Bmatrix} \rightarrow \left(\frac{1}{3}\right)^6 = \frac{1}{729}$$

$2\ 2\ 2\ 2 \rightarrow 1$

So, the final cases and their probabilities are:

$$(2, 0, 4) \rightarrow \frac{1}{729} + \frac{2}{729} + \frac{3}{729} + \frac{4}{729} + \frac{5}{729} = \frac{15}{729} = \frac{5}{243}$$

$$(2, 1, 3) \rightarrow \frac{4+8+12+16+20}{729} = \frac{60}{729} = \frac{20}{243}$$

$$(2, 2, 2) \rightarrow \frac{90}{729} = \frac{10}{81}$$

$$(2, 3, 1) \rightarrow \frac{60}{729} = \frac{20}{243}$$

$$(2, 4, 0) \rightarrow \frac{1+2+3+4}{729} + \frac{1}{324} + \frac{2}{243} = \frac{73}{2916}$$

Subcase III. There is one 0 in the sequences.

a. The zero is in the first position

0 _ _ _ _ _

$1\ 1\ 1\ 1\ 2 \rightarrow \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{1}{486}$

$$\left.\begin{matrix} 11121 \\ \dots \\ 21111 \end{matrix}\right\} \rightarrow 4\cdot\left(\frac{1}{3}\right)^6 = \frac{4}{729}$$

$1\ 1\ 1\ 2\ 2 \rightarrow 10\cdot\left(\frac{1}{3}\right)^6 = \frac{10}{729}$

$1\ 1\ 2\ 2\ 2 \rightarrow 10\cdot\left(\frac{1}{3}\right)^6 = \frac{10}{729}$

$1\ 2\ 2\ 2\ 2 \rightarrow 5\cdot\left(\frac{1}{3}\right)^6 = \frac{5}{729}$

$2\ 2\ 2\ 2\ 2 \rightarrow \left(\frac{1}{3}\right)^6 = \frac{1}{729}$

The sequence for cases b., c., and d. are similar (second, third and fourth position).

e. The zero is in the fifth position:

_ _ _ _ 0 _

$1\ 1\ 1\ 1\ \_\ 2 \rightarrow \left(\frac{1}{3}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{324}$

$$\left.\begin{matrix} 1111\_2 \\ \dots \\ 2111\_1 \end{matrix}\right\} \rightarrow 4\cdot\left(\frac{1}{3}\right)^6 = \frac{4}{729}$$

$1\ 1\ 1\ 2\ \_\ 2 \rightarrow \frac{10}{729}$

$1\ 1\ 2\ 2\ \_\ 2 \rightarrow \frac{10}{729}$

$1\ 2\ 2\ 2\ \_\ 2 \rightarrow \frac{5}{729}$

$2\ 2\ 2\ 2\ \_\ 2 \rightarrow \frac{1}{729}$

f. The zero is in the sixth position:

$\_\ \_\ \_\ \_\ \_\ 0$

$1\ 1\ 1\ 1\ 2 \rightarrow \frac{1}{324}$

$$4 \cdot \begin{Bmatrix} 1\,1\,1\,2\,1 \\ \ldots \\ 2\,1\,1\,1\,1 \end{Bmatrix} \rightarrow \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{1}{486}$$

$1\ 1\ 1\ 2\ 2 \rightarrow \frac{10}{729}$

$1\ 1\ 2\ 2\ 2 \rightarrow \frac{10}{729}$

$1\ 2\ 2\ 2\ 2 \rightarrow \frac{5}{729}$

$2\ 2\ 2\ 2\ 2 \rightarrow \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{1}{486}$

So, the final cases and their probabilities are:

$$(1,0,5) \rightarrow 4 \cdot \frac{1}{729} + \frac{1}{729} + \frac{1}{486} = \frac{13}{1458}$$

$$(1,1,4) \rightarrow 4 \cdot \frac{5}{729} + \frac{5}{729} + \frac{5}{729} = \frac{30}{729} = \frac{10}{243}$$

$$(1,2,3) \rightarrow 4 \cdot \frac{10}{729} + \frac{10}{729} + \frac{10}{729} = \frac{20}{243}$$

$$(1,3,2) \rightarrow 4 \cdot \frac{10}{729} + \frac{10}{729} + \frac{10}{729} = \frac{60}{729} = \frac{20}{243}$$

$$(1,4,1) \rightarrow 4\left(\frac{4}{729} + \frac{1}{486}\right) + \frac{1}{324} + \frac{4}{729} + \frac{1}{324} + \frac{2}{243} = \frac{73}{1458}$$

Subcase IV. There are no 0s in the sequences.

$\_\ \_\ \_\ \_\ \_\ \_$

$$1\ 1\ 1\ 1\ 2\ 2 \rightarrow \left(\frac{1}{3}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{324}$$

$$\left.\begin{matrix} 111212 \\ \dots \\ 211112 \end{matrix}\right\} \rightarrow 4 \cdot \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{2}{243}$$

$$2\ 2\ 1\ 1\ 1\ 1 \rightarrow \frac{5 \cdot 4}{2} \cdot \left(\frac{1}{3}\right)^6 = \frac{10}{729}$$

$$1\ 1\ 1\ 2\ 2\ 2 \rightarrow \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \left(\frac{1}{3}\right)^6 = \frac{20}{729}$$

$$1\ 1\ 2\ 2\ 2\ 2 \rightarrow \frac{6 \cdot 5}{2} \cdot \left(\frac{1}{3}\right)^6 = \frac{15}{729} = \frac{5}{243}$$

$$\left.\begin{matrix} 122222 \\ \dots \\ 222212 \end{matrix}\right\} \rightarrow \frac{5}{729}$$

$$2\ 2\ 2\ 2\ 2\ 1 \rightarrow \left(\frac{1}{3}\right)^5 \cdot \frac{1}{2} = \frac{1}{486}$$

So, the final cases and their probabilities are:

$$(0,4,2) \rightarrow \frac{1}{324} + \frac{2}{243} + \frac{10}{729} = \frac{73}{2916}$$

$$(0,3,3) \rightarrow \frac{20}{729}$$

$$(0,2,4) \rightarrow \frac{5}{243}$$

$$(0,1,5) \rightarrow \frac{5}{729} + \frac{1}{486} = \frac{13}{1458}$$

The final probability is:

$$P=2\left(\frac{233}{5832}\cdot\frac{73}{2916}+\frac{233}{1944}\cdot\frac{20}{729}+\frac{233}{1944}\cdot\frac{5}{243}+\frac{233}{5832}\cdot\frac{13}{1458}+\frac{5}{243}\cdot\frac{73}{1458}+\frac{20}{243}\cdot\frac{23}{243}+\frac{10}{81}\cdot\frac{20}{243}+\frac{20}{243}\cdot\frac{10}{243}+\frac{73}{2916}\cdot\frac{13}{1458}\right)\cdot 100=$$

$$=100\cdot 2\cdot\frac{487843}{17006112}=\frac{487843}{8503056}\cdot 100\sim 5.73726669564448358...$$

So, **P~5.73%.**

We also made a code in C++ for this part of the problem which works for the general case.

Remark: For "big numbers" it may have problems when running due to complexity.

In order to have access to the code, please go to the following github repository:

https://github.com/ProjectMej/MEJ.git

4. CONCLUSION

In order to solve this problem, we have used probability and combinatorics. For the first part, we calculated the number of favorable paths and the number of total paths. We analyzed some basic particular cases: the case in which the grid is a segment - where we obtained a probability of 0% if the strictly positive coordinate is odd and a probability of 100% if the strictly positive coordinate is even and a case in which the grid is square shaped – case $(0,2,2)$, where we obtained a probability of 37.5%. We have proved that in order for the bots to be able to meet, the sum of the coordinates of the grid has to be even, otherwise they will pass one another.

Then, we worked on the case when the grid has coordinates $(3,4,5)$, obtaining a probability of ~5.73%.