Snowflakes

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Insight into the mathematical world

"If we look upon it the right way, not only does math hold the truth, but it also represents the supreme beauty – a cold, austere beauty, the same as a sculpture's." (Bertrand Russell, 1918, "Mysticism and Logic")

"The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. Maybe it is hard to define the beauty of math, but it is equally hard to define any kind of beauty – we do not know why a poem is beautiful, but we discover its beauty by reading it." (G.H. Hardy, 1940, "A Mathematician's Apology")

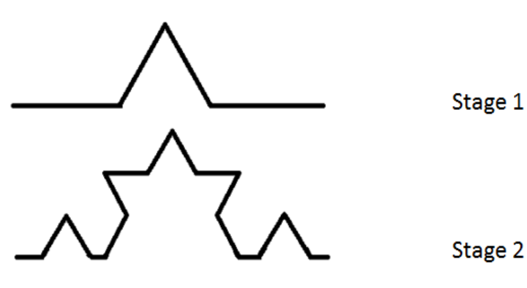
The coastline paradox

If the coastline of Great Britain is measured using units of 100 km long, then the length of the coastline is approximately 2,800 km. With 50 km units, the total length is approximately 3,400 km (approximately 600 km longer).

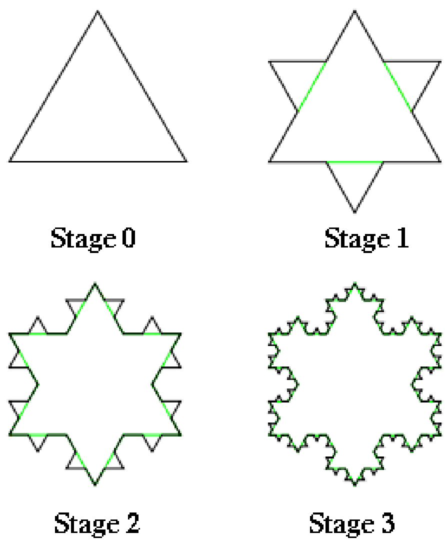
Koch's snowflake

An idealized mathematical analogy of the coastline is obtained by taking a look at a geometrical shape , first considered by H. Von Koch in 1904, called Koch snowflake.

In order to create the Koch Snowflake, von Koch developed the Koch Curve.



The Koch Curve starts with a straight line, which is then divided into 3 equal parts. Using the middle segment as a base, an equilateral segment is created. Finally, the base of the triangle is removed.



The picture describes the Koch Snowflake in four different stages.

Stage 0 depicts the Koch snowflake seen out of rocket.

From such distance it looks exactly like an equilateral triangle. The closer we get to Earth, the clearer it gets that each of the 3 sides presents another equilateral triangle, positioned on the middle third of a side, as shown in Stage 1.

If the perimeter of the shape in Stage 0 is of 3 units, the perimeter in Stage 1 will be $3 \times \frac{4}{3}$ units.

Getting even closer, we notice that each of the, now, 12 segments presents an equilateral triangle, positioned on the middle third of a segment.

Stage 2: Now the perimeter is $3 \times \frac{4}{3}$ x $\frac{4}{3}$ units.

Stage 3 describes the shape seen much closer, suggesting the possible true shape of the Koch snowflake.

On the whole, Koch snowflake is built by starting with an equilateral triangle, removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely.

The exact mathematical perimeter of the Koch Snowflake is the curve of the sequence of an infinite number of approximations. We attain this curve by following a similar process to the one in which the sequence of an infinite number of decimal approximations approaches 1/3.

0,3 , 0,33, 0,333, 0,3333, 0,33333, ...

The Koch Snowflake has a well defined area. While its numerical value is in strong connection with the units in which it is measured, it will definitely be finite.

As a conclusion, after each iteration, the perimeter increases by a factor 4/3. While we are reaching the Koch Curve, the factor of 4/3 appears infinitely many times. Therefore the length of the Koch Curve is infinite.

Sierpinski's Carpet

To build the Sierpinski carpet you take a square, cut it into 9 equal-sized smaller squares, and remove the central smaller square. Then you apply the same procedure to the remaining 8 subsquares, and repeat this *ad infinitum*.

Our task - *I*

Each one of the sides of an equilateral triangle (with a side of 1 u.m.) is divided in three equal parts. The middle segment is replaced by an equilateral triangle and afterwards the original segment is erased. This way, we will obtain a polygon with twelve equal sides. For each side of the new polygon we repeat the mentioned process and thus we obtain what we call an iterative process.

a) *We consider the obtained polygon after ten iterations. How many sides does it have? Calculate its perimeter. Generalize for any given number of iterations. What do we notice when n (number of iterations) becomes a very large number?*

For the number of sides, let $(a_n)_{n\geq 0}$ the sequence which defines the number of sides after *n* iterations.

$a_0 = 3 = 3\times 2^0$

$a_1 = 12 = 3\times 2^2$

$a_2 = 48 = 3\times 2^4$

...

$a_n = 3\times 2^{2n}$ (P(n))

We prove that $a_n=3\times 2^{2n}$ by using mathematical induction:

I. $n=0 \Rightarrow a_0 = 3\times 2^0 =3$ (true)

II. Assume P(n) holds. Then, it must be shown that P(n+1) holds.

$a_n = 3\times2^{2n} \Rightarrow a_{n+1} = 3\times2^{2n} \times 2^2$ (q.e.d., because two segment

are added to each side)

III. Since I and II hold, the
statement P(n) holds for all natural numbers n. (q.e.d.)

$a10 = 3\times2^{20} = 3145728$

As for the perimeter, let $(b_n)_{n\geq0}$ the sequence which defines the perimeter of

the shape after *n* iterations.

$b_0 = 3$

$b_1 = 12 \times 1/3 = 3\times2^2 \times 1/3$

$b_2 = 48\times(1/3)^2 = 3\times2^4 \times (1/3)^2$

....

$b_n = 3\times2^{2n}\times(1/3)^n$

This formula, $b_n = 3\times2^{2n}\times(1/3)^n$, can be proved by using

mathematical induction.

$b_{10} = 3\times2^{20}\times(1/3)^{10} = 3\times2^{20}/3^{10} = 3\times4^{10}/3^{10} = 4^{10}/3^9 = 104$

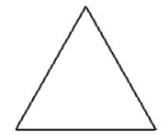
$8576/19683 \approx 53{,}273179$

b) Calculate the area of the obtained polygon after 10 iterations.

Generalize for any given number of iterations. What do we notice when n becomes a very large number?

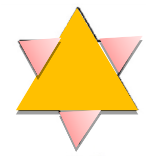
If $A=\frac{l^2\sqrt{3}}{4}$ is the area of any equilateral triangle with the side of length *l*

For l=1=> $c_0 =\frac{\sqrt{3}}{4}$, the area of the initial triangle.



$A_1=\frac{\left(\frac{1}{3}\right)^2\sqrt{3}}{4}$ (the area of the )

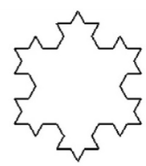
Using a) and multiply it by a_0 ($a_n =3\cdot 2^{2n}$), which is equal to 3



n = 1

$c_1 = c_0 + a_0 \cdot A1 =\frac{\sqrt{3}}{4}+ 3\cdot\frac{\left(\frac{1}{3}\right)^2\sqrt{3}}{4}$=> $c_1 = \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3})$

We use the same method for the next number in the sequence



n = 2

$c_2 = c_1 + a_1 \cdot A2= \frac{\sqrt{3}}{4}\cdot\left[1 + \frac{1}{3} + 3\cdot 2^2\left(\frac{1}{3^2}\right)^2\right] = \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3})$

We continue to do that until we notice something we can use to find a form for c_n .

$c_3 = \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5})$

We notice that to the "brackets" we add $\frac{2^{2(n-1)}}{3^{2(n-1)+1}}$. So, using the mathematical induction, we will demonstrate that

P(n): $c_n = \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2(n-1)}}{3^{2(n-1)+1}})$., where $n\geq1$.

I For n=1=> $C_1 =\frac{\sqrt{3}}{4}(1+\frac{2^{2(1-1)}}{3^{2(1-1)+1}})=\frac{\sqrt{3}}{4}(1+\frac{1}{3^1})$, which is true.

II Supposing that P(n) is true, we demonstrate that P(n+1) is true as well

If $c_n = \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2(n-1)}}{3^{2(n-1)+1}})$

then $c_{n+1}= \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2n}}{3^{2n+1}})$

$c_n + \frac{\sqrt{3}}{4}\cdot(\frac{2^{2n}}{3^{2n+1}})= \frac{\sqrt{3}}{4}\cdot(1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2n}}{3^{2n+1}})= c_{n+1}$

III Conlusion:

Since I and II hold => $c_n = \frac{\sqrt{3}}{4} \cdot (1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2(n-1)}}{3^{2(n-1)+1}})$

We notice that the sum includes a geometric progression, so we solve it using the well known sum: $S=b1\cdot\frac{1-q^n}{1-q}$, where q= ratio, n= number of fractions and b1= first number in the sequence.

$S = \frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{18}}{3^{19}} = b1\cdot\frac{1-q^n}{1-q} = \frac{1}{3}\cdot\frac{1-(\frac{2^2}{3})^n}{\frac{-1}{3}} = \frac{1}{3}\cdot 3\cdot\frac{(\frac{2^2}{3})^n-1}{1} = \frac{2^{2n}}{3^n}-1$

Using this, we will determine $c_n = \frac{\sqrt{3}}{4} \cdot (1+\frac{1}{3}+\frac{2^2}{3^3}+\frac{2^4}{3^5}+\ldots+\frac{2^{2(n-1)}}{3^{2(n-1)+1}}) = \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\cdot\frac{1}{3}\cdot\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right)) =$

$\frac{\sqrt{3}}{4}\cdot\left[1+\frac{1}{3}\cdot\frac{\left(\frac{4}{9}\right)^n-1}{\frac{4}{9}-1}\right] = \ldots = \frac{\sqrt{3}}{4}\cdot\left[\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{2}{3}\right)^{2n}\right]$

So $c_{10} = \frac{\sqrt{3}}{4}\cdot\left[\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{2}{3}\right)^{20}\right]$=0,6927.

In order to find out what happens when „n" gets very high, we must calculate the limit of (C_n):

$$\lim_{n\to\infty}\left[\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\cdot\frac{1}{3}\cdot\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right] = \frac{\sqrt{3}}{4}\cdot\frac{8}{5}$$

$C_0 < Cn < 2\cdot C_0$

c) What surprising conclusion do we reach after solving a) and b)?

After a considerable number of iterations, we notice that the area obtained is finite, while the perimeter is infinite.

Our task – *II*

Each one of the sides of an equilateral triangle (with a side of 1 m.u.) is divided in 3 equal parts. The middle segment is replaced by an equilateral triangle towards the interior and afterwards the original segment is erased.

Similarly to point (I), after an iterative process we obtain different interesting shapes.

Calculate the area of the obtained shape after 6 iterations. Generalise for a random number of iterations. What do we notice when n becomes a large number?

Let $(d_n)_{n\geq 0}$ the sequence which defines

the perimeter of the shape after *n*

iterations.

$$d_0 = 3 = 2^0 \times 3 \; sides$$

$$d_1 = 12 = 2^2 \times 3 \; sides$$

$$d_2 = 48 = 2^4 \times 3 \; sides$$

Therefore, the figure has the same number of sides as the one from part I and the area:

$$C'_n = \frac{\sqrt{3}}{4} \cdot \left(1 - \frac{1}{3} - \frac{2^2}{3^3} - \frac{2^4}{3^5} - \ldots - \frac{2^{2(n-1)}}{3^{2(n-1)+1}}\right)$$

$$C'_n = \frac{\sqrt{3}}{4} \cdot \left[1 - \frac{1}{3} \cdot \frac{\left(\frac{4}{9}\right)^n - 1}{\frac{4}{9} - 1}\right] = \ldots = \frac{\sqrt{3}}{4} \cdot \left[\frac{2}{5} + \frac{3}{5} \cdot \left(\frac{2}{3}\right)^{2n}\right]$$

$$C'_6 = \frac{\sqrt{3}}{4} \cdot \left[\frac{2}{5} + \frac{3}{5} \cdot \left(\frac{2}{3}\right)^{12}\right] = 0{,}174$$

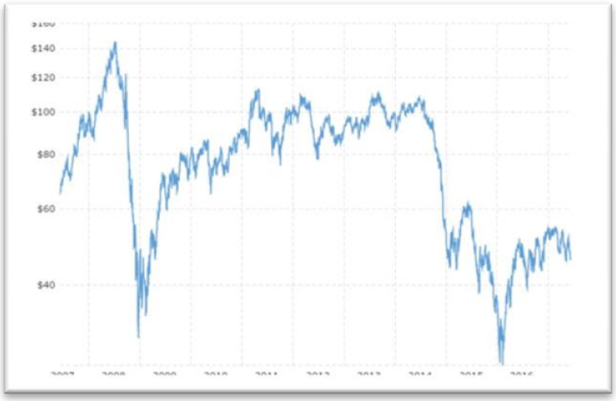
Fractals in daily life

A neat geometrical form with irregular outline, which can be divided into pieces and each piece is a smaller replica of the original is what we call a fractal.

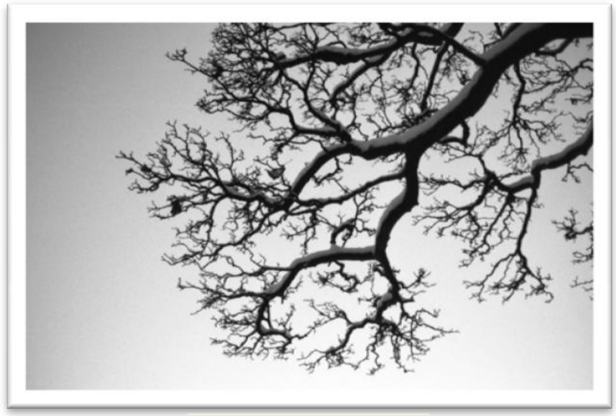
Benoit Mandelbrot characterized them as „Beautiful, heavy, more and more useful".

Even in economy, we can imagine the evolution of a price over the time as a fractal.

Fractals in art and nature are aestethically pleasing and stress-reducing, due to the presence of repetitive patterns. Thinking “out of the box” leads to unexpected but potentially revolutionary ideas.



Example of evolution of a price


Tree branches


Parque Natural de Doñana, Spain


Britain’s coast