On Time

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Our team has completed a set of tasks related to a classic clock:

a) How many degrees lie between the hour hand and the minute hand when the clock shows 4:32?

b) Which is the first moment between 7 and 8 o'clock when the angle determineted by the hour hand and the minute hand measures 100^0?

c) Suppose the clock stops randomly, which is the probability that the hour hand lies between 7 and 10 o'clock?

d) A snail travels clockwise, 23 minutes a day. In its travel, the snail pulls after him the minute hand and the hour hand of a clock. On this special clock, the snail starts from the 12th hour. After how many days will the snail stop in the position when a normal clock shows 51 minutes?

After how many days will the snail stop in the position in which the minute hand and the hour hand of this clock show 11:51 o'clock on a normal clock?

e) The hand hour of an office clock is 3 cm long and the minute hand is 4 cm long. In random choosed moments during a day it was noticed that the triangle formed by these two hands has its sides of integer lenght. Calculate the probability that the triangle is isosceles.

f) Let's suppose we can switch the hour hand with the minute hand on a regular clock. In which situations the new hour is correctly indicated? (the hour indicated by hands makes sense)

g) Ana left home in the morning, between 7 and 8 o'clock, and she returned, between 16 and 17 o'clock. She notices that the minute hand and the hour hand were in the same positions, but switched between them. Estimate the hour she left home.

a) The clock is divided in 12 intervals and each interval is divided in 5 subintervals. It means that we have 60 subintervals. Let "X" be the number of degrees of a small interval: $X = 360° /60 = 6°$.
If within 30 minutes the hour hand moves 15 °, then within 2 minutes it will move 1 °.

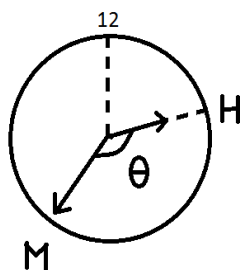At 4:30 the hour hand and the minute hand form a 45° angle. In 2 minutes the minute hand moves 12°.

The measure of the angle formed by the hour hand and the minute hand at 4:32 is $45° - 1° + 12° = 56°$.

Remark: We can set a general formula to determine the angle formed by the hour hand and the minute hand at a certain moment.

The hour hand runs 360º in 12 hours => 720 minutes, so the speed is $v_h = 0.5°$/minute. The minute hand runs $360°$ in 60 minutes, so $v_m = 6°$/ minute.

Let "Θ" be the searched angle at H.M (H-hour, M-minute). Until this moment, there will have passed $60 \cdot H + M$ minutes (starting at 12 o'clock) and the angle between 12:00 and the hour hand will be $\Theta_h = 0.5 \cdot (60 \cdot H + M)$. The angle between 12:00 and $\Theta_m = 6° \cdot M$ and $\Theta = |\Theta_m - \Theta_h|$ or $\Theta = |0.5° \cdot (60 \cdot H + M) - 6° \cdot M| => |30° \cdot H - 5.5° \cdot M| = \Theta$.

If the hour is 4:32 (H = 4, M = 32), then $\Theta = |0.5° \cdot (60 \cdot 4 + 32) - 6° \cdot 32| = 56$.

b₁) First of all, we have to find out how many degrees the minute hand is moving in a minute. If it moves $15°$ in 30 minutes, then in one minute it will move $0.5°$.

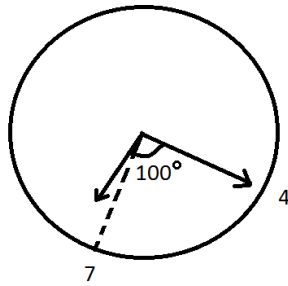Also, we know that the minute hand is moving $6°$ per minute.

At 7 o'clock there are 7 intervals between the hour hand and minute hand and this means $210°$.

The angle between the minute hand and the hour hand will decrease with $6° - 0.5°$/minute. So, from $210°$ to $100°$, we have $110°$ that will decrease after $110° : 5.5°$/minute = 20 minutes.

This means that at 7:20 the angle between the hour hand and the minute hand will be $100°$.

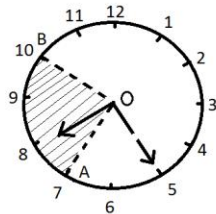*b₂)* We use the formula that we set at a), switching $\Theta = 100°$:
$100° = |30°H - 5.5°M|$, where H = 7. It follows that $210° - 5.5°M = \pm 100°$. Since we are searching for the minimum of M, the solution that works is M = 20 and the unknown hour is 7:20.

c) To determine this probability, we cannot use the ratio between the number of favorable situations and the number of possible situations, because the hour hand moves an infinity in the highlighted zone.

We will use the geometric probability, that means:

P = the highlighted area/the area of the disk = the area of the circle section ∢AOB/the area of the disk = measure of ∢AOB/360^0 = $30^0 \cdot 3/360^0$ = 1/4.

Historical note:

Georges-Louis Leclerc, Comte de Buffon (1707 – 1788) was the first mathematician who introduced the probability calculus in geometry. He first made his mark in the field of mathematics with the problem of *Buffon's needle* (named after him). Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips? Buffon's needle was the earliest problem in geometric probability to be solved. The solution, in the case where the needle length is not greater than the width of the strips, can be used to design a Monte Carlo method for approximating the number π. 100 years later it was found that the probability is $\frac{2l}{\pi a}$, *l* = the length of the needle and *a* = the width of the parallel strips.

d_1) We can say that in one day the snail, traveling clockwise, goes through 23 subintervals. In 2 days it goes through 46 subintervals, and in 3 days he goes through 69 subintervals. So, by going through the clock dial the snail does not stop where the minute hand of a regular clock shows 51 minutes. On the third day, the snail moves at the same time with the dial and it starts all over again (from 12 or 0 o'clock). The snail stops (at the end of the day) at the 9th subinterval.

The equation is: $n·23 = 51 + 60·c$, where n = the number of days, c = the number of complete laps. Multiplying by 13 and $23·13 = 299 = 300−1 = 60·5−1 => c = 21$ and n = 57.

In conclusion, the minimum number of days in which the snail reaches the moment when the minute hand shows 51 minutes is 57.
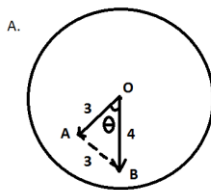
d_2) After 57 days the snail and the hour hand of this clock show 21:51(9:51) o'clock on a regular clock, because c = 21. When the snail's lap is completed (the minute hand's lap) the hour hand moves through an interval = 5 subintervals. Let be c' = $11+12 · k$, k = positive integer, k ≥ 2. From the previous equation we obtain n = 657. That means the snail reaches 11:51 sharp after 657 days. (!!!)

e) Suppose that at a certain moment, the minute hand, the hour hand and the distance between them, form a triangle with the lengths of the sides positive integers. Since the lengths of the sides in any triangle are a, b, c the following relations hold: $|b-c| <a < b + c$, $|c-a| < b < c + a$ and $|a-b| < c < a + b$, we get the number of favorable situations for the distance between the arms :{ 2, 3, 4, 5, 6}. The isosceles triangle can have two sides of the length of 3 or 4, so the number of favorable cases is {3, 4}. So the probability is $\frac{2}{5} = 0,4$. We can establish the moments when this happens.
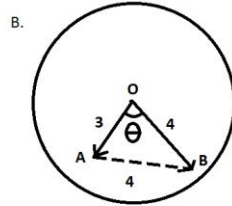
In the first situation (A) we calculated cos Θ = $(OA^2+ OB^2−AB^2): (2·OA·OB) = 3^2+4^2−3^2:2·3·4 = ⅔$, so Θ ≅ 48°. Since Θ =$|30°·H − 5,5°M| => 30°−5,5°M = ± 48° =>$ 60H−11M = 96 => M = (60H± 96) :11 ∈ ℕ, where H ∈ ℕ, H ≤ 12 .

From 11/60H ± 96 =>11/5H ±8 => H ∈ {5, 6}.

In conclusion the possibilities are 5:36 and 6:24.

In the second situation (B) Θ ≅ 68° and 30°H−5, 5°M = ± 68° we get (60H±136):11 ∈ ℕ=> H ∈{3, 8} and the hours are 3:04 and 8:56.

f) We suppose it is: H: M o'clock, Hϵ {0,1, ... ,11}, M ∈ {0,1, ... ,59}.

At H:M o'clock, the minute hand and the 12th hour form an angle of M·6^0 and the hour hand with the 12th hour another angle of H·5·6^0+M·5/60·6^0.

We switch the minute hand with the hour hand and we suppose that, after the change, the clock shows "a right" hour, that means H': M'.

Now the minute hand and the 12th hour form an angle:

H'·6^0 = H·5·6^0 + $\frac{M·5}{60}$ · 6^0 ⇔

H'·5·6^0+$\frac{M'·5}{60}$ = M· 6^0 ⇔

⇔ M'= H·5+ M/12

⇔ H'·5+M'/12=M

It follows: 12·M'=60·H+M and 60·H'+M'=12·M

or 13· $(M' - M) = 60 · (H - H')$ and   11· (M' + M) $= 60 · (H + H')$,

H + H'=11/60·(M + M') < 22

Since 0 ≤ H, H' ≤ 12, there are 12·12−1 = 143 possibilities.

We can represent in a coordinate axis system the multitude of pairs (x, y) representing the positions of the two hands on the dial. We are interested in those pairs (x, y) for which the pairs (y, x) also belong to this multitude. But the point (y, x) is symmetrical with point (x, y). The number of pairs is to be found at the intersection of the lines in the figure, meaning 143.

g) In the equality established at *f)* we replace H = 7, H'= 4 and then:
$60 \cdot 7 + M = 12M'$, $60 \cdot 4 + M' = 12M \Rightarrow M' \cong 37$ and $M \cong 23$.
Ana left at 7:23 and came back at 4:37.

Bibliography:

1) V. BOBANCU, *Caleidoscop matematic*, Albatros Publishing House, Bucharest, 1979
2) *Probleme de matematica traduse din revista KVANT*, Didactic and Pedagogical Publishing House, Bucharest 1983
3) Google images
4) *Mathworld.wolfram.com*
5) Wikipedia.org
6) *Sotruefacts.com/everydaylife*

Note d'édition :

Cet article, bien construit, bien écrit et clair, a toute sa place pour ces raisons d'être publié sur le site de Math.en.Jeans et témoigne d'un gros travail de la part des élèves. Il ne doit pas cependant être pris pour modèle de production d'atelier Math.en.Jeans car le type de sujet qui a été proposé à cet atelier n'est pas un sujet Math.en.Jeans puisque il laisse peu de place à l'autonomie des élèves, et ne leur permet pas d'aborder ce travail comme le ferait un chercheur en mathématiques.