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The Mathematics of Paper Folding

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Research Topic Description

This project describes some of the constructions which are possible by making multiple folds of a piece of paper.

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1 Statement of the problem

We know how to construct lengths with a ruler (with no gradations) and a compass and we also know that some lengths cannot be constructed. But what numbers can we construct through foldings? For example, if we had a string of paper of length 1, how could we fold it in order to have a string of paper of length $\frac{1}{3}$?

2 Results

In this paper we will present our approach for the construction of strings of paper of arbitrary positive rational length. We will also take a closer look at the structure the constructible lengths of strings form, by observing possible extensions of our problem. Some of the possible paper folding constructions can also be put in direct comparison to the straightedge and compass constructions, showing that in fact paper folding constructions allow more freedom than straightedge and compass.

3 Defining our problem

We are going to model the folds of the paper through geometric constructions and therefore, we look for specific geometric elements which can be constructed.

Definition 3.1. A line l is constructible if a fold can be made along l.

Definition 3.2. A point A is constructible if there are two constructible lines that intersect at A.

Definition 3.3. A number α is constructible if there are two constructible points at distance α .

3.1 Notations

In our approach, we have used the following sets to allow us to define the rules of construction.

- \mathscr{L} = the set of constructible lines in the cartesian plane
- \mathscr{P} = the set of constructible points in the cartesian plane
- \mathscr{C} = the set of constructible numbers

3.2 Assumptions

In our research we use some reasonable assumptions to be able to extend our research topic. Hence, we begin our construction with two points in the cartesian plane. We define the length of the segment between these points to be our unit distance. We assume that our sheet of paper is unbounded (such that we are able to work with the full cartesian plane). In our construction we will also allow a segment of length 0 (which can be obtained using two overlapping points) and allow negative elements in \mathscr{C} .

3.3 Constructions axioms

Similar to the axioms of geometry, we use several rules (1) which define the kind of folds allowed. In our research we will only be allowed to make one fold at a time and we have to unfold the piece of paper before being able to fold it again.

Axiom 1. Given two points A and B, we can fold a line connecting them.

Axiom 2. Given two points A and B, we can fold A onto B (Perpendicular bisector).
Axiom 3. Given two lines l₁ and l₂, we can fold l₁ onto l₂ (Angle Bisector).



Axiom 4. Given a point A and a line l, we can make a fold that is perpendicular to l and passes through A.

Axiom 5. Given two points A and B and two lines l_1 and l_2 , we can make a fold that places A onto l_1 and B onto l_2 (Translating a segment).

Axiom 6. Given two points A and B and a line l, we can make a fold that passes through A and places B on l.



3.4 Geometry Theorems used

Theorem 3.1 (Right Triangle Altitude Theorem). If h denotes the altitude in a right triangle and p and q the segments determined by the altitude on the hypotenuse then the following relation is true:

$$h^2 = pq.$$

3.5 Defining Abstract Algebra concepts used

Definition 3.4 (Monoid). A monoid is a set M together with an operation "*" on M which satisfies two axioms:

(i) "*" is associative, meaning that (x * y) * z = x * (y * z), for any three elements from M

(ii) There is an element e in M such that x * e = e * x = x, for every x in M

Definition 3.5 (Group). A group is a set G together with an operation "*" on G which satisfies three axioms:

- (i) "*" is associative, meaning that (x * y) * z = x * (y * z), for any three elements from G
- (ii) There is an element e in G such that x * e = e * x = x, for every x in G
- (iii) Each element x of G has an inverse $x' \in G$ such that x * x' = x' * x = e

Definition 3.6. An abelian group G is a group such that for all $x, y \in G$ we have that x * y = y * x

Definition 3.7 (Ring). A ring is a set R together with two operations (usually $+, \cdot$) which satisfies the following conditions:

- (i) Under addition R is an abelian group
- (ii) Under multiplication R is a monoid
- (iii) For all $x, y, z \in R$ we have

x(y+z) = xy + yz and (y+z)x = yx + zx.

Definition 3.8 (Field). A commutative ring such that the subset of nonzero elements form a group under multiplication is called a field.

4 Building the cartesian plane

We could answer some specific questions about constructible lengths without using many difficult concepts, but in order to find a way to describe the elements in \mathscr{L} , \mathscr{P} and \mathscr{C} it would be helpful if we could use concepts from analytical geometry, such as denoting points by their cartesian coordinates and lines by their equations.

We denote the two initial points by O and I and we will consider O to be the origin of the plane and I to be of coordinates (1,0). By using Axiom 1 we construct the line connecting O and I (which is actually the x-axis).

We are going to define a sequence of folds such that we can obtain a point J that has the following properties: $JI \perp OI$ and JI = OI

Lemma 1. Given two points $A, B \in \mathscr{P}$ we can construct a point C such that $AC \perp AB$ and AC = AB.

Proof. • Using Axiom 1 we construct line l_1 passing through A and B.

- Using Axiom 4 we construct line l_2 perpendicular on l_1 in B.
- Using Axiom 4 we construct line l_3 perpendicular on l_1 in A.
- Using Axiom 3 we construct line l_4 , the bisector of the right angle formed by l_1 and l_2 . Let $\{C\} = l_3 \cap l_4$.

The length of AC is equal to the length of AB and $AC \perp AB$.



Figure 7: Lemma 1

We are going to prove that the intersection of lines l_3 and l_4 is actually the point we needed. Since the angles formed by l_4 with l_1 and l_3 are both 45° we have that ΔBAC is an isosceles rightangled triangle, so AC = AB and $AC \perp AB$. (Note that we have two possible constructions for point C. We can choose either one.)

Using Lemma 1 we can build both the y-axis and point J on this axis, such that OJ = 1.

5 Constructing the integers

To define a way to prove that $\mathbb{Z} \times \mathbb{Z} \subset \mathscr{P}$, we will need to use another lemma.

Lemma 2. Given two points $A, B \in \mathscr{P}$ we can construct a point A', such that A' is the reflection of A through B.

Proof. By applying Lemma 1 twice we can successively obtain point C such that CB = AB and $CB \perp AB$ and then point $A'(\neq A)$ such that A'B = CB and $A'B \perp BC$.



Figure 8: Lemma 2

Because both $AB \perp BC$ and $A'B \perp BC$, we have that A, B, A' are collinear and AB = BC = A'B. Hence, A' is the reflection of A through B.

Returning to our problem, we can now use the lemma above to construct all points of integer coordinates. Firstly, we construct the points of integer coordinates on the x-axis and y-axis, by applying Lemma 2 multiple times to create consecutive segments of length 1. Afterwards, to construct the point $(m, n), m, n \in \mathbb{Z}$ we can use Axiom 4 twice, to create perpendiculars on the axis through points $(m, 0) \in \mathscr{P}$ and $(0, n) \in \mathscr{P}$. The intersection of these perpendiculars will be point (m, n).

6 Constructing the rationals

After finding a way to easily construct points with integer coordinates, we want to see if we can move on to points of rational coordinates. It turns out that all rational numbers are also constructible and we will now provide two methods for constructing them.

6.1 First Method

Rationals of the form $m/2^n$

The first observation is that we can easily obtain rational numbers with denominator a power of 2. To obtain the fraction $\frac{1}{2}$ we can use *Axiom* 1 and *Axiom* 2 to build the intersection of the perpendicular bisector of segment (*OI*) and *x*-axis. Let M_1 be this intersection. Then, $OM_1 = \frac{1}{2}$. To obtain k/2, $k \in \mathbb{Z}$ we can use a process similar to the one we described for obtaining the integers.



Arbitrary rational number

Firstly, we will show that it is possible to construct the fraction $\frac{1}{3}$ and then by induction we will show that it is possible to extend our algorithm to any fraction of the form $\frac{1}{n}$, $n \in \mathbb{N}$. Using the following two lemmas we will be able to prove the induction step.

Lemma 3. The fraction $\frac{1}{3}$ is constructible.

Proof. We start our proof by constructing point J as described in the previous sections. Now, by applying Axiom 4 twice, to construct perpendiculars from I on x-axis and from J on y-axis. Let K(1,1) be the intersection of these perpendicular. Using the following set of constructions we can get to a segment of length $\frac{1}{3}$:

• Using Axiom 2 we construct l_1 , the perpendicular bisector of (OI).

Let $\{M\} = l_1 \cap OI$ be the midpoint of (OI).

- Using Axiom 2 we construct l₂, the perpendicular bisector of (KM).
 Let {N} = l₂ ∩ KM.
- Using Axiom 1 we construct l_3 , the line connecting M and N.
- Using Axiom 4 we construct l₄, the perpendicular in M on l₃.
 Let {P} = l₄ ∩ OJ.
- Using Axiom 2 we construct l₅, the perpendicular bisector of (PO).
 Let {Q} = l₅ ∩ PO be the midpoint of (PO).



Figure 11: Folding $\frac{1}{3}$

Because $N \in l_2$, we have that KN = NM = x and KN + NI = 1. We also have that $NI \perp MI$ and $MI = OM = \frac{1}{2}$, so by Pythagoras's Theorem we get that:

$$x^{2} = (1-x)^{2} + \frac{1}{4} \implies x = \frac{5}{8} \implies NI = \frac{3}{8}$$

Because $\mu(\measuredangle NMP) = 90^\circ$, we have that $\mu(\measuredangle NMI) + \mu(\measuredangle PMO) = 90^\circ$. Since $NI \perp MI$, we also have that $\mu(\measuredangle NMI) + \mu(\measuredangle MNI) = 90^\circ$, so $\measuredangle MNI \equiv \measuredangle PMO$. Similarly, $\measuredangle NMI \equiv \measuredangle MPO$. Hence, the triangles $\triangle NMI$ and $\triangle MPO$ are similar ($\triangle NMI \sim \triangle MPO$). We have obtained the following relation:

$$\frac{PO}{MI} = \frac{OM}{NI} \implies \frac{PO}{\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{8}} \implies PO = \frac{2}{3}.$$

Because $PQ = QO = \frac{PO}{2} = \frac{1}{3}$, we get that $\frac{1}{3} \in \mathscr{C}$.

Lemma 4. If fraction $1/k \in \mathscr{C}$, then $1/(k+1) \in \mathscr{C}$.

Proof. We are going to start by constructing point K(1,1) as described in Lemma 3. Because $1/k \in \mathscr{C}$ we can construct point $M \in (OI)$ such that MI = 1/k. Similarly to the previous lemma we define the following set of constructions:

- Using Axiom 2 we construct l₂, the perpendicular bisector of (KM).
 Let {N} = l₂ ∩ KM.
- Using Axiom 1 we construct l_3 , the line connecting M and N.
- Using Axiom 4 we construct l₄, the perpendicular in M on l₃.
 Let {P} = l₄ ∩ OJ.
- Using Axiom 2 we construct l_5 , the perpendicular bisector of (PO). Let $\{Q\} = l_5 \cap PO$ be the midpoint of (PO).

By applying Pythagoras' Theorem in ΔNMI we can easily get that (2)

$$NI = \frac{(k-1)(k+1)}{2k^2}$$

Because $\Delta NMI \sim \Delta MPO$ we get that:

$$\frac{PO}{MI} = \frac{OM}{NI} \implies \frac{PO}{\frac{1}{k}} = \frac{\frac{k-1}{k}}{\frac{(k-1)(k+1)}{2k^2}} \implies PO = \frac{2}{k+1}$$

Because PQ = OQ = PO/2 = 1/(k+1), we get that $1/(k+1) \in \mathscr{C}$.

We have now proved that every fraction 1/n, $n \in \mathbb{N}$ is constructible, so we can apply Lemma 2 multiple times to construct every fraction m/n, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ (the same process used to construct the integers).

6.2 Second Method

Lemma 5. Given $\alpha, \beta \in \mathcal{C}$, we also have that $\alpha\beta \in \mathcal{C}$.

Proof. We begin our proof by constructing x-axis and y-axis and point O(0,0) as described before.

- Since 1, α , $\beta \in \mathscr{C}$, we can construct points J(0,1), $A(\alpha,0)$, $B(0,\beta)$.
- Using Axiom 1 we construct the line connecting J and A.
- Using Axiom 4 we construct line l_1 perpendicular to AJ.
- Using Axiom 4 we construct line l_2 perpendicular to l_1 . Let $\{C\} = OA \cap l_2$.

We have that $JA \parallel BC$. Since $\Delta OAJ \sim \Delta OCB$, we get that

$$\frac{OC}{OA} = \frac{OB}{OJ} \Rightarrow \frac{OC}{\alpha} = \frac{\beta}{1}$$

Hence $OC = \alpha\beta$ and $\alpha\beta \in \mathscr{C}$.

Lemma 6. Given $\alpha \in \mathscr{C}, \alpha \neq 0$, we also have that $1/\alpha \in \mathscr{C}$

• Since $1, \alpha \in \mathscr{C}$ we begin by constructing the x-axis and y-axis and points O(0,0), I(1,0), $A(0,\alpha)$.

- Using Axiom 1 we construct the line connecting A and I.
- Using Axiom 4 we construct line l perpendicular to AI. Let $\{B\} = l \cap AO$.

Using the Right Triangle Altitude Theorem we get that $IO^2 = BO \cdot AO$ and so $BO = 1/\alpha$. Hence $1/\alpha \in \mathscr{C}$.

We have proven in the previous sections that $\mathbb{N} \subset \mathscr{C}$, so we can apply the two lemmas above to construct any rational number $m/n, m, n \in \mathbb{N}$. Using Lemma 6 we have that $1/n \in \mathscr{C}$ and then by using Lemma 5 for 1/n, m we get that $m/n \in \mathscr{C}$.



7 Structure of \mathscr{C}

7.1 Field Structure

Lemma 7. Given α, β constructible numbers, we can also construct $\alpha + \beta$ and $\alpha - \beta$.

Proof. Construct points $A(\alpha, 0)$, $B(-\beta, 0)$. The length $AB = \alpha + \beta$.

Construct points $A(\alpha, 0)$, $B(\beta, 0)$. The length $AB = \alpha - \beta$. (Assuming we allow negative lengths if $\alpha < \beta$).

In both cases we could use Axiom 5 to translate segment AB to the origin of the cartesian plane.

Hence $\alpha + \beta, \alpha - \beta \in \mathscr{C}$. (Choosing $\alpha = 0$ we get that $-\beta \in \mathscr{C}$ for all $\beta \in \mathscr{C}$.)

We have now seen that \mathscr{C} is closed under addition and multiplication and that $(\mathscr{C}, +)$ is a group (with $0 \in \mathscr{C}$ as the additive identity) and $(\mathscr{C} \setminus \{0\}, \cdot)$ is also a group (with $1 \in \mathscr{C}$ as the multiplicative identity). We can now conclude that \mathscr{C} has a field structure with $\mathbb{Q} \subset \mathscr{C}$.

7.2 Taking square roots

Lemma 8. Given $\alpha \in \mathscr{C}$, we can also construct $\sqrt{\alpha}$ (3).

• We begin by constructing x - axis, y - axis and points O(0,0), $A(\alpha,0)$.

- Using Lemma 5 and Lemma 6 we get that $(\alpha k)/2 \in \mathcal{C}$, for all $k \in \mathcal{C}$. Take $0 < k < \alpha$ and construct point $B((\alpha k)/2, 0)$.
- Using Axiom 6 construct line l_1 placing A on the y-axis, passing through B.
- Using Axiom 4 construct line l₂ perpendicular to l₁, passing through A.
 Let {A'} = l₂ ∩ y − axis and let x = OA'.



Figure 14: Folding $\sqrt{\alpha}$

The length $OA'=\sqrt{k\alpha}:$ using Pythagoras' Theorem for right-angled triangle $\Delta BOA'$ we get that

$$\left(\frac{\alpha+k}{2}\right)^2 = \left(\frac{\alpha-k}{2}\right)^2 + OA'^2 \Rightarrow OA' = \sqrt{k\alpha}.$$

Therefore, taking k = 1 (4), we get that $\sqrt{\alpha} \in \mathscr{C}$.

In the sections above we have proven that \mathscr{C} is actually closed under simple operations like addition and multiplication and that it even has a field structure. Here, we have also proven that the set of constructible numbers is closed under taking square roots. This seems to be a very powerful statement, since it shows that the set of constructible numbers using straightedge and compass is actually a subset of the set of constructible numbers using paper folding (5).

8 Trisecting an acute angle

It is well known that the classical straightedge and compass problem of trisecting an angle has been proven to be unsolvable. However, we can see that using paper folding we can find a way to trisect an acute angle, showing that in fact paper folding constructions may even be more powerful that straightedge and compass constructions.

Lemma 9. Given an acute angle θ with vertex in O(0,0), we can construct two lines trisecting angle θ .

Proof. We assume that θ is formed between the x-axis and an arbitrary line l that intersects the origin.

- Take $k \in \mathscr{C}$. Since 2k is also in \mathscr{C} , we can construct points Q(0,k) and P(0,2k).
- Construct the *y*-axis.
- Using Axiom 4 construct the perpendicular l_1 from Q on y-axis.
- Using Axiom 5 construct a fold that places P on l and O on l_1 . Mark this fold by l_2 .
- Using Axiom 4 construct line l_3 such that $P \in l_3$ and $l_3 \perp l_2$. Let $\{P'\} = l \cap l_3$.
- Using Axiom 4 construct line l_4 such that $O \in l_4$ and $l_4 \perp l_1$. Let $\{O'\} = l_1 \cap l_4$.
- Using Axiom 1 construct line l_5 connecting P' and O'.
- Using Axiom 4 construct line l₆, the perpendicular from O on P'O'.
 Let {Q'} = l₆ ∩ P'O'.

Lines OQ' and OO' trisect angle θ .



Figure 15: Trisecting an angle

Since l_2 is the perpendicular bisector of PP' and OO' we have that PP'O'O is an isosceles trapezoid. Therefore, PO' = P'O ($\Delta POO' \equiv \Delta P'O'O$).

Since PQ = QO and $QO' \perp PO$, $\Delta POO'$ is isosceles, so PO' = OO'. Hence, OO' = OP'. Now we have that $\Delta OP'Q' \equiv \Delta OO'Q$ (since P'O = O'O, OQ' is a common edge and $\mu(\measuredangle P'Q'O) = \mu(\measuredangle O'Q'O) = 90^{\circ}$). Therefore

$$\measuredangle P'OQ' \equiv \measuredangle O'OQ' \quad and \quad P'Q' = Q'O' = \frac{P'O'}{2} = k.$$

Let *M* be the projection of *O'* on the *x*-axis, then O'M = k. Moreover, we also have that $\Delta OO'M \equiv \Delta OO'Q'$, so $\angle O'OM \equiv \angle O'OQ'$.

Hence $\mu(\measuredangle P'OQ') = \mu(\measuredangle O'OM) = \mu(\measuredangle O'OQ') = \theta/3$ and we have construct two lines that trisect an arbitrary angle.

9 Conclusion and further research

We have closely observed the structure of the set of constructible numbers discovering that it is closed under common operations like addition and multiplication, that it has a field structure and that it is even closed under taking square roots. Folding paper has proven to be more powerful than straightedge and compass, allowing us to solve problems unsolvable using only straightedge and compass.

However, our research on this interesting set of numbers can be extended even further. We could ask ourselves what is the complete characterization of \mathscr{C} , trying to discover what other kinds of operations can be applied on \mathscr{C} (such as taking cube, fourth and fifth power roots).

We could also think about the possible extensions of our problem. We could allow two or even more folds at the same time, which would enable constructions impossible using only the axioms we have described in section 3.

10 References

- [1] https://www.youtube.com/watch?v=SL21YcggGpc
- [2] https://www.youtube.com/watch?v=6Lm9EHhbJAY

Notes d'édition

(1) These basic rules for paper folding are known as Huzita–Justin axioms or Huzita–Hatori axioms.

(2) The proof is the same as that of Lemma 3, except that here MI = 1/k: with the same notation, Pythagoras' Theorem yields $x^2 = (1 - x)^2 + 1/k^2$, so $2x = 1 + 1/k^2$ and $NI = 1 - \frac{1}{2}(1 + 1/k^2) = (k^2 - 1)/2k^2$.

Below, similarity of the triangles ΔNMI and ΔMPO is proven exactly as in Lemma 3.

(3) Of course, we have to assume $\alpha \ge 0$ here, and even $\alpha > 0$ for the proof below.

(4) This presupposes $\alpha > 1$, but otherwise we can choose k = 1/n with some $n > 1/\alpha$, construct $\sqrt{k\alpha} = \sqrt{\alpha/n}$ and apply Lemma 5 with $\sqrt{\alpha/n}$ and \sqrt{n} to get $\sqrt{\alpha} \in \mathscr{C}$ (or construct $\sqrt{1/\alpha}$ and apply Lemma 6).

(5) Indeed, it is known that numbers which are constructible using straightedge and compass are exactly those which are obtained from rational numbers by taking sequences of square roots and field operations (Wantzel's Theorem), so they all belong to \mathscr{C} . Note that it is not yet proven that \mathscr{C} is strictly larger, but this will be a consequence of the possibility of trisecting the angles as explained in the next section.