Irrational numbers

2020-2021

Students: Ștefan Strugariu, Bogdan Duminică, Matei Dumitrescu, Alexandru Ciocoiu, Iulia Bilan
School: National College Iasi, Romania
Teacher: Gabriel Popa
Researcher: Claudiu Volf, "Alexandru Ioan Cuza" University Iasi, Romania

PROBLEM STATEMENT

Let $d \neq 1$ be an integer. Determine all numbers of the form $a + b\sqrt{d}, a, b \in \mathbb{Z}$, such that their inverses are of the same form ($a' + b'\sqrt{d}, a', b' \in \mathbb{Z}$). We assume that d is square-free ($d \neq 0$ and 1 and d is not divisible with the square of any prime; d can be 2, 3, -1, 6, …, but not 8 or 18).

THE MAIN IDEA

Let $d \in \mathbb{Z}$ be a square-free integer and let us define the set $M_d = \{x = a + b\sqrt{d} \ / a, b \in \mathbb{Z}\}$. We define the **norm** of *x* by the function $N(x) = a^2 - db^2 \in \mathbb{Z}$. We will prove that $x \in M_d$ has an **inverse** in M_d if and only if $N(x) \in \{-1, 1\}$.

Proof:

Let $x \in M_d, x = a + b\sqrt{d}, a, b \in \mathbb{Z}$. We define the **conjugate** of *x* by $\bar{x} = a - b\sqrt{d}, a, b \in \mathbb{Z}$.
We may easily verify that $N(x) = x \cdot \bar{x}$. Moreover, it is trivial to prove that:

$\overline{xy} = \bar{x} \cdot \bar{y}$; $N(\bar{x}) = N(x)$

The function $N(x)$ is **multiplicative**, since for every $x, y \in M_d$ we have:

$$N(xy) = xy \cdot \overline{xy} = x\,y\,\bar{x}\,\bar{y} = x\,\bar{x}\,y\,\bar{y} = N(x) \cdot N(y)$$

Suppose that $x \in M_d$ has an inverse in M_d, hence there exist $y \in M_d$ such that $xy = 1$. We may write:

$$1 = N(1) = N(xy) = N(x) \cdot N(y)$$

We conclude that *N(x)* admits an inverse and, moreover, $N(x) \in \{-1, 1\}$.

Conversely, if $N(x) = \pm 1$, then $x \cdot \bar{x} = \pm 1$, hence $\pm \bar{x} \in M_d$ is the inverse of *x*.

As a consequence, integers *a* and *b* verify the equation $a^2 - db^2 = \pm 1$.

SPECIAL CASE : d = -1

In this case we have $\sqrt{d} = i$, the imaginary unit number. Let $T = \frac{1}{a+b\sqrt{d}}$, $a, b \in \mathbb{Z}$. Because T is a part of the set M_d, there must exist integers x and y such that $T = x+y\sqrt{d}$. Now we get: $\frac{1}{a+b\sqrt{d}} = x+y\sqrt{d} \Rightarrow \left(a+b\sqrt{d}\right)\left(x+y\sqrt{d}\right) = 1$. We notice that both terms in the left side of the equation are complex numbers, giving us a hint towards a good way to approach it.

By applying the modulus operation to both sides and using the multiplicative property of the modulus, we get:

$$\left|a+b\sqrt{d}\right|\cdot\left|x+y\sqrt{d}\right| = 1 \Rightarrow |a+b\cdot i|\cdot|x+y\cdot i| = 1 \Rightarrow \sqrt{a^2+b^2}\cdot\sqrt{x^2+y^2} = 1$$

By squaring both sides we finally get $(a^2+b^2).(x^2+y^2) = 1,\ a, b, x, y \in \mathbb{Z}$. Because the square of an integer is a natural number, we conclude that the only possible solutions for the d = -1 case are $i, -i$, 1 and -1.

SPECIAL CASE : d = 2

The ancient Greeks were among the first to study the equation $a^2 - 2b^2 = 1$ motivated by the evaluation of the diagonal of a unit square that equals the irrational number $\sqrt{2}$. As such, they generated arbitrarily close approximations of this number by using arbitrarily large solutions of the equation above $a_n^2 - 2b_n^2 = 1$, since we may write:

$$\frac{x_i^2}{y_i^2} = 2 + \frac{1}{y_i^2} \xrightarrow[y_i \to +\infty]{} 2 \quad (1)$$

Hence, if y_i denotes the side of a square, x_i represents its diagonal. Defining the "side" and "diagonal" numbers s_i and d_i by the relations:

$$\begin{matrix} d_1 = 3;\ s_1 = 2 & & d_1^2 - 2s_1^2 = 1 \\ & \Rightarrow & \\ d_{i+1} = d_i + 2s_i\ ;\quad s_{i+1} = d_i + s_i & & d_{i+1}^2 - 2s_{i+1}^2 = -\left(d_i^2 - 2s_i^2\right) \end{matrix}$$

we conclude that **odd-indexed** numbers $\left(d_i, s_i\right)$ represent solutions of the equation $a^2 - 2b^2 = 1$, while **even-indexed** pairs verify the closely related equation $a^2 - 2b^2 = -1$.

Let us further analyze the equations above, by searching integer numbers $a, b \in \mathbb{Z}$ such that $a^2 - 2b^2 = \pm 1$. When 1 appears in the right-hand side we obtain a **positive Pell equation**, while for -1 we get a **negative Pell equation**.

> ***Remark:*** Since every (a, b) solution of the Pell equations yields additional solutions of the form
>
> $(a, -b), (-a, b), (-a, -b)$, it will be sufficient to search for *natural* values of the a and b parameters.

To start with, we remark that $y_1 = 3 + 2\sqrt{2} \in M_2$ has the norm equal to 1 and according to the proof above it is a solution of the associated Pell equation. Then the number: $y_2 = y_1^2 = \left(3+2\sqrt{2}\right)^2 = 17+12\sqrt{2}$ has also norm 1, because:

$$N(y_2) = N(y_1\cdot y_1) = N(y_1)\cdot N(y_1) = 1$$

Along the same line of thought, any power of the form $y_n = y_1^{\,n}$ has also norm 1 and will also represent a solution of the corresponding Pell equation. Moreover, we may prove by induction that such powers are distinct. More specifically, if $y_1^n = u + v\sqrt{2}$, $u,\ v \in \mathbb{N}$, then:

$y_1^{\,n+1} = y_1^{\,n} \cdot y_1 = \left(u + v\sqrt{2}\right)\cdot\left(3+2\sqrt{2}\right) = \left(3u+4v\right) + (2u+3v)\sqrt{2}$

and since $3u+4v > u$, $2u+3v > v$ we conclude that $y_1^{\,n}$ and $y_1^{\,n+1}$ are distinct. Hence, the Pell equation for *d* = 2 has an infinite number of solutions. (2)

The number $y_1 = 3 + 2\sqrt{2}$ is called the **fundamental solution** of the *positive* Pell equation, that is the non-trivial solution having the smallest value (the trivial solution of the equation is obtained for *a* = 1 and *b* = 0). Much similar, the number $1+\sqrt{2}$ represent the fundamental solution of the *negative* Pell equation.

The general solution of the positive Pell equation $a^2 - 2b^2 = 1$ is obtained from the relation: $a_n + b_n\sqrt{2} = \left(a_0 + b_0\sqrt{2}\right)^n$, where $\left(a_n, b_n\right)$ satisfy the recursive relation below, and $\left(a_0, b_0\right) = \left(3, 2\right)$ is the fundamental solution:

$$a_{n+1} = a_0 a_n + 2b_0 b_n$$
$$b_{n+1} = b_0 a_n + a_0 b_n$$

The general solution of the negative Pell equation $a^2 - 2b^2 = -1$ is indicated by the recursive relations below:

$$a_{n+1} = a_0 x_n + 2b_0 y_n$$
$$b_{n+1} = b_0 x_n + a_0 y_n$$

where $\left(a_0, b_0\right) = \left(1, 1\right)$ represents its fundamental solution, while $\left(x_n, y_n\right)$ is the general solution of the associated *positive* Pell equation.

THE GENERAL CASE

We will first consider the **positive** equation $a^2 - db^2 = 1$. In this case, since *d* is not a perfect square, we can find an infinite number of solutions. We denote by $\left(a_0, b_0\right)$ the fundamental solution of the equation (smallest solution different from (1, 0)). We denote the sequence:

$$a_1 = a_0\ ;\ b_1 = b_0$$
$$a_{n+1} = a_0 a_n + db_0 b_n$$
$$b_{n+1} = b_0 a_n + a_0 b_n$$

Solving this sequence we find the solutions:

$$a_n = \frac{1}{2}\left[\left(a_0 + b_0\sqrt{d}\right)^n + \left(a_0 - b_0\sqrt{d}\right)^n\right]\ ;\ b_n = \frac{1}{2\sqrt{d}}\left[\left(a_0 + b_0\sqrt{d}\right)^n - \left(a_0 - b_0\sqrt{d}\right)^n\right]$$

The **negative** equation $a^2 - db^2 = -1$ **does not always have solutions** (*e.g.*, for *d* = 3). Much similar to the positive case, the general solution (when one exists) is given by the relation $a_n \pm b_n\sqrt{d} = \left(a_0 \pm b_0\sqrt{d}\right)^{2n+1}$, where $\left(a_0, b_0\right)$ denotes the fundamental solution. The general solutions defined above for the positive equation are valid also for the negative one, except that only odd values of *n* are permitted.

Conclusion

The proposed problem enabled us to get a deeper understanding of the theory behind the celebrated Pell's equation. We additionally found that this equation is related to square-triangular numbers (perfect squares that also represent Gaussian sums) and continuous fraction representation of quadratic irrational numbers. As such, we convinced ourselves once again that Gauss was right when stating that "*Mathematics is the queen of the sciences and number theory is the queen of mathematics*".

Notes d'édition

(1) The ratings must be simplified and the use of terms "side" and "diagonal" seems irrelevant.
So we propose :
In order to approximate $\sqrt{2}$, the ancient Greeks used arbitrarily large solutions of the equation $x^2 - 2y^2 = 1$ and if x and y are such that x^2 − 2 y^2 = 1 and y is arbitrarily large, we have :

$$\frac{x^2}{y^2} = 2 + \frac{1}{y^2} \underset{y \to \infty}{\longrightarrow} 2$$

Hence $\frac{x}{y}$ is an approximation of $\sqrt{2}$.

(2) The result has not been demonstrated, it has been accepted.