Half-plane geometry

2018-2019

Students: Simone Andreolli, Lucia Menetto, Giacomo Scuttari

School: Liceo Scientifico "Romano Bruni", Padova, Italy

Teachers: Dario Benetti, Lina Falcone

Researchers: prof. Francesco Rossi, prof. Alberto Zanardo (Università degli studi di Padova)

Abstract

Consider a half-plane delimited by a line h, where lines are semicircles centred on h. We show what happens to polygons.

Contents

(1)

1 Introduction: the problem

The text of the problem says:

Let us consider as a "plane space" a half-plane delimited by a line h, as "lines" the semicircles of center on h and as "points" those of the half plane, we obtain a geometry that verifies the axioms of classical Euclidean geometry with the exception of the postulate V (axiom of parallel lines).

What happens to polygons? For example, the sum of the angles is no longer constant!

2 Background information

First of all, basing on the text of the problem, we define the primitive notions of our geometry:

- plane: half-plane delimited by an Euclidean line called h, which does not belong to the plane;
- point: it corresponds to the Euclidean point;
- line: Euclidean semicircle centered on h;
- half-line: each of the two parts in which a line is divided by a point O called origin, non-belonging to h;
- degenerate line: Euclidean half-line perpendicular to h with its origin on h. It is a line with infinite radius;
- segment: part of a line defined by two distinct points A and B, not belonging to h;
- polygon: figure consisting of consecutive segments;
- angle: each part of the plane identified by two half-lines of the same origin, including the two half-lines.

Secondly, Let us study the five axioms of the Euclidean geometry to be able to create a geometry that verifies them all, except for the fifth. In particular, not having any metric and so any way to define length, we decide to investigate the Hilbert axioms instead of the Euclidean axioms (see [1]). Following the axioms (2) :

1. Given two distinct points A and B there is one and only one line that contains them both.
2. Each line r is an ordered set of points, such that taken on r two distinct points A and B there is always a point C of r between A and B.
3. There are infinite lines for each point. On a line passing through A and B there are two possible orders on r, the one for which A precedes B and the one for which B precedes A.
4. Given two oriented lines a and b and two points $A \in a$ and $B \in b$ then there are two rigid motions that lead a to coincide on b, with A on B; one makes the two orientations coincide and the other arranges them in the opposite direction.

5. Given a line a and a point $P \notin a$, there exists one and only one line b containing P parallel to a (thus, a and b are disjoint).

The first axiom is verified by the Euclidean definition of half-circle, except in the case where the two points are positioned vertically. For this reason it is necessary to introduce the degenerate lines. The second axiom is verified by the Euclidean definitions of half-line and line. To verify the third and the fourth axioms it is necessary to create an appropriate rigid motion. It will be analyzed in the next section. To deny the fifth axiom let us define as parallel lines all the lines which do not intersect each other. In this way there are an infinite number of possible lines b containing P parallel to a.

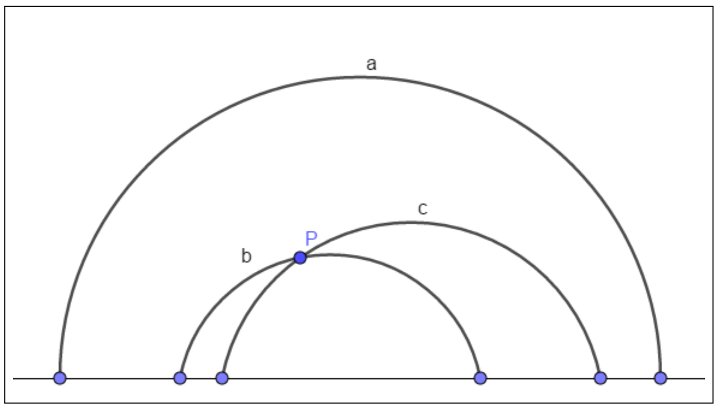

Figure 1

At this point, in order to be able to answer the question of the problem, we need a way to prove if two segments are congruent, or if they are not, which one is bigger and which one is smaller. Thus, we have to find a rigid motion which makes us able to do that and which respects all the axioms except for the fifth.

3 The rigid motion

In this section we analyse the rigid motion. Given a segment on a line, the rigid motion will enable us to create a congruent segment on another line. As a consequence, it makes us able to overlap two different segments and so to proof if two different segments are congruent or not. In this way we are able to create polygons.

After several attempts, we decided to ask for help from the professor Franz Rothe from University of North Carolina at Charlotte. Together we developed a rigid motion which respects all the preconditions above and guarantees that the congruence of segments is an equivalence relation. So we have the reflexive, symmetric and transitive property (see [2]).

The reflexive and symmetric properties are obvious. Instead, acquiring what the professor Franz Rothe suggested, we accept the transitive property, even if we are not able to prove it, in order to go on with the problem.

To better explain the rigid motion, we divide it into five paragraphs (3).

3.1 From a line to a different line

In this paragraph we analyze the rigid motion which, given a segment on a line and a point on a different line, makes us able to create a congruent segment on the second line starting from the given point on it. The new segment can be created on both orientations relative to the point on the second line.

Referring to Figure 2, let CD be a segment on the line c and E the point on the line d. Let us start from c and create two Euclidean half-lines (we call them g and s) starting from an endpoint of the line c and passing through the two endpoints C and D of the segment. Let us create the Euclidean half-line (we call it l) starting from an endpoint of d (which has to be on the opposite side of the endpoint of the first line) and passing through the point E. In this way we find two intersections between the Euclidean half-lines. Let us call them G and I. Let us create a degenerate line passing through G. As a consequence we find another intersection, F, with s. Let us create an Euclidean half-line starting from the endpoint of d and passing through F. We find a point L on d. The segment LE is congruent to the initial segment CD. Let us do the same for the second intersection, I. We find another segment on d. It is congruent to CD and by the transitive property it is also congruent to LE.

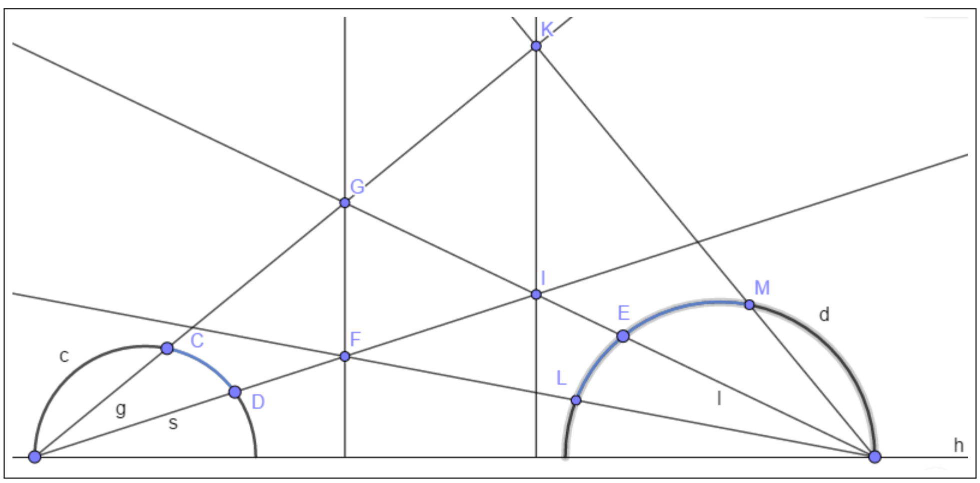

Figure 2

3.2 From a line to the same line

In this paragraph we analyze the rigid motion which, given a segment on a line and a point on the same line, makes us able to create a congruent segment on the line starting from the given point on it. The new segment can be created on both orientations relative to the point on the line.

Referring to Figure 3, let CD be a segment on a line and E the point on the same line. Let us start creating the two Euclidean half-lines g and s starting from the endpoint B and passing trough the two endpoints of the segment CD. Let us create the Euclidean half-line l starting from the endpoint A of the line and passing through E. In this way we find two intersections between the Euclidean half-lines. Let us call them F and G. Let us create a degenerate line passing through F. As a consequence we find the intersection H with s. Let us create an

Euclidean half-line starting from A and passing trough H. We find a point L on the line. The segment LE is congruent to CD.

Let us do the same for the second intersection, G. We find the segment EM; it is congruent to CD and, by the transitive property, it is also congruent to LE.

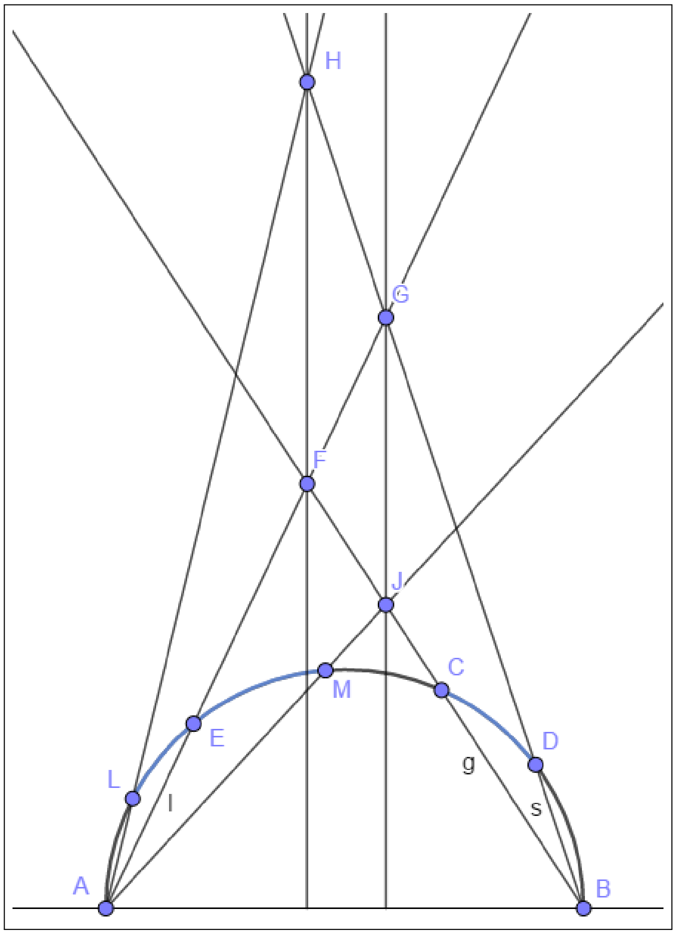

Figure 3

3.3 From a line to a degenerate line

In this paragraph we analyze the rigid motion which, given a segment on a line and a point on a degenerate line, makes us able to create a congruent segment on the degenerate line starting from the given point on the degenerate line. The new segment can be created on both orientations relative to the point on the degenerate line.

Referring to to Figure 4, let CD be a segment on a line a and E a point on the degenerate line l. Let us create the two Euclidean half-lines g and s starting from the endpoint A of the line and passing through the two endpoints of the segment. Let us create a Euclidean line i parallel to h passing through E. In this way we find two intersections between the Euclidean half-lines g and s and the Euclidean line i. Let us call them F and G. Let us create the degenerate line passing through F. As a consequence we find the intersection H with s. Let us create the Euclidean line p parallel to h passing through H. We find the point M on the intersection between l and p. The segment ME is congruent to the initial segment CD. Let us do the same for the second intersection, G. We find the segment EL. It is congruent to CD and, by the transitive property, it is also congruent to ME.

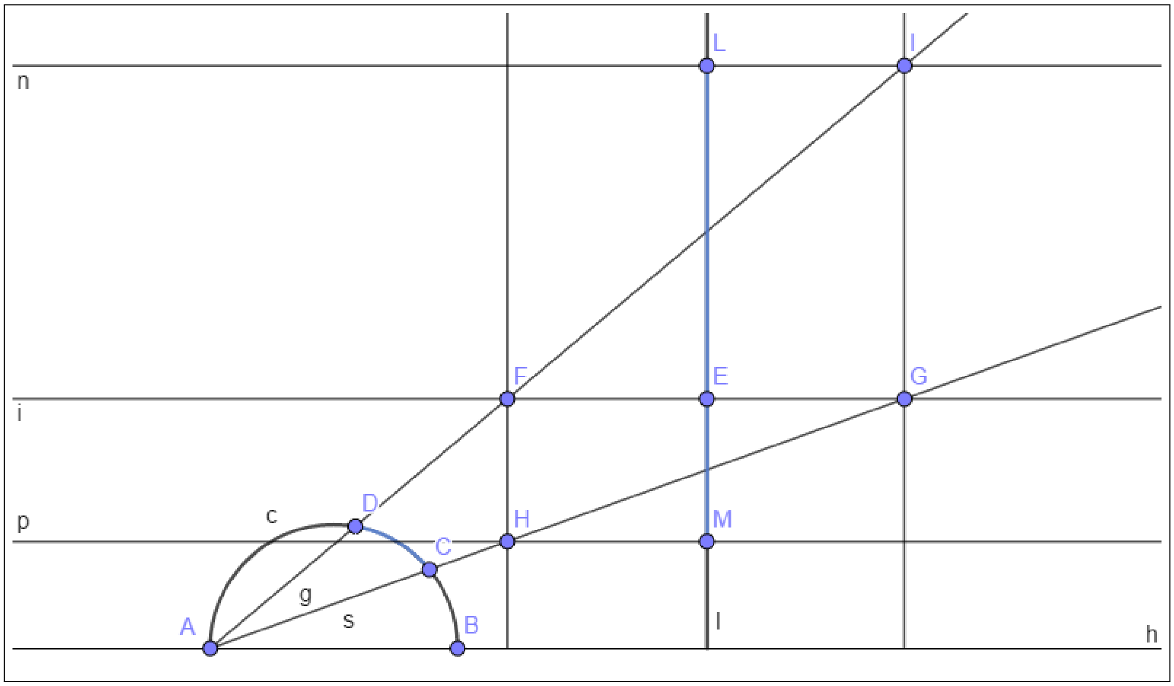

Figure 4

3.4 From a degenerate line to a different degenerate line

In this paragraph we analyze the rigid motion which, given a segment on a degenerate line and a point on a different degenerate line, makes us able to create a congruent segment on the second degenerate line starting from the given point on it. The new segment can be created on both orientations relative to the point on the degenerate line.

Referring to Figure 5, let DC be a segment on a degenerate line t and E a point on another degenerate line c. Let us create the Euclidean line g passing through E and D. It intersect h in the point Q. Let us create the Euclidean half-line m starting from Q and passing through C. We find the point L on the intersection between c and l. The segment LE is congruent to DC. Let us to the same creating the Euclidean line s passing through E and C. We find the segment ME. It is congruent to DC and, by the transitive property, it is also congruent to LE.

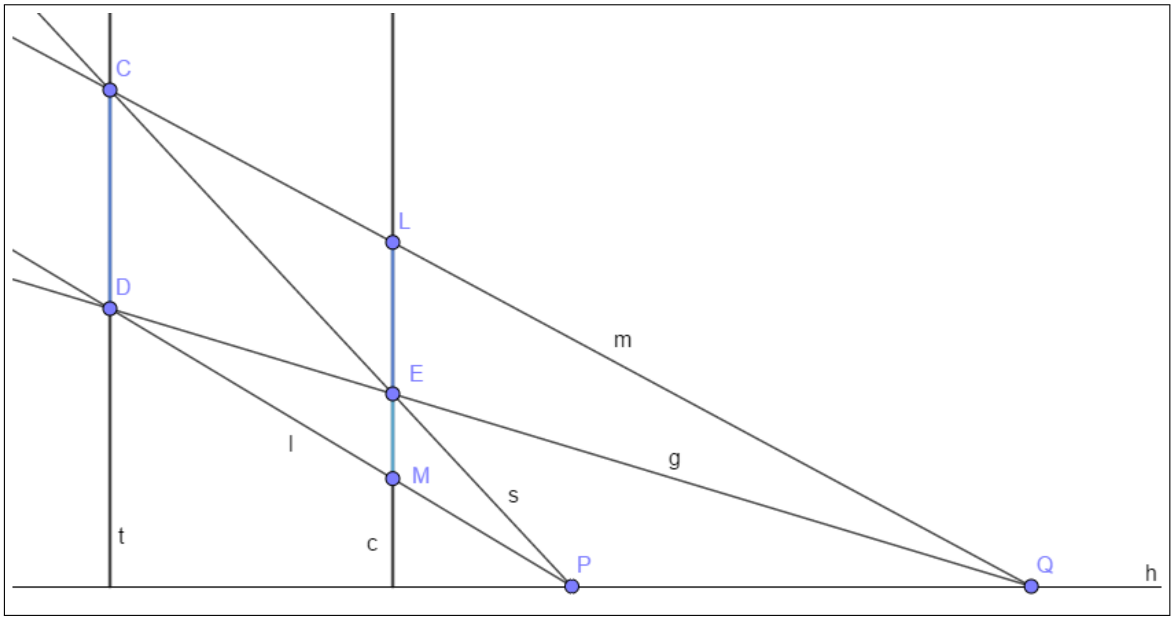

Figure 5

3.5 From a degenerate line to the same degenerate line

In this paragraph we analyze the rigid motion which, given a segment on a degenerate line and a point on the same degenerate line, makes us able to create, a congruent segment on the degenerate line starting from the given point on it. The new segment can be created on both orientations relative to the point on the degenerate line.

Referring to Figure 6, let CD be a segment and E a point on a degenerate line c. Let us create a degenerate line t on the half-plane. Then let us create the two Euclidean lines q and p parallel to h passing through the two endpoints of the segment CD. We find a segment HI on t which is congruent to CD. Now we have t with the segment HI and the point E on c. Let us use the rigid motion [3.4]. We find the two segments ME and EL which are both congruent to the initial segment CD.

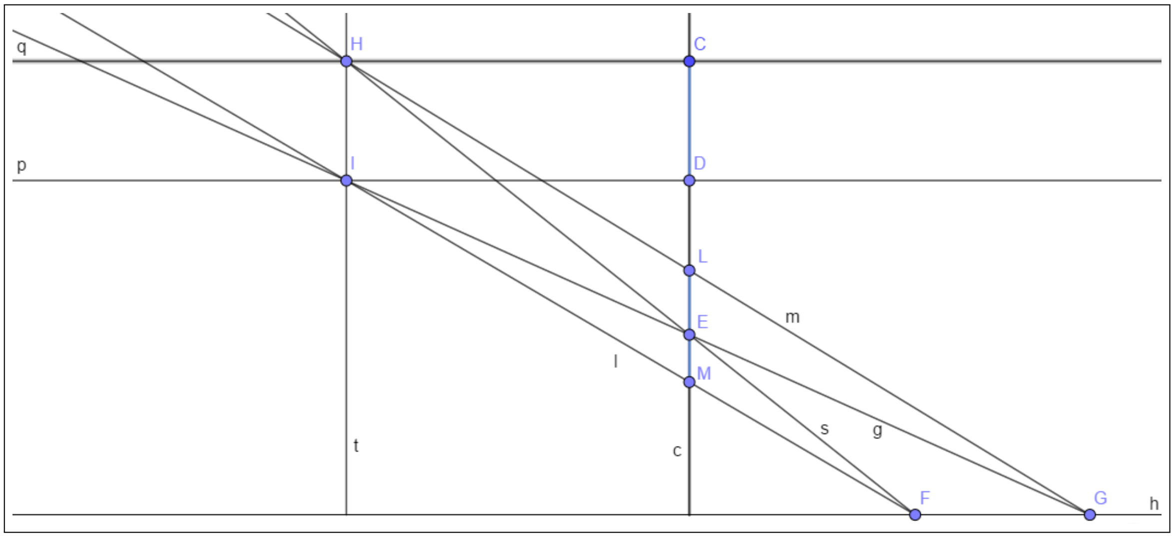

Figure 6

4 Polygons

4.1 Definition

A polygon is a figure delimited by at least three consecutive segments, called sides. The angle of a polygon is the plane space between two consecutive sides. In order to measure an angle let us create the Euclidean tangents to the lines passing through the origin of the angle. We measure the angle between the two tangents in the Euclidean way.

In the next paragraphs we will study the following polygons:

- Triangle; that is a polygon delimited by three sides;
- Quadrilateral; that is a polygon delimited by four sides;
- Regular polygon with more than four sides; that is a polygon delimited by at least five congruent segments called sides.

4.2 Sum of the interior angles of triangles

To better understand the next study of triangles we start with the analysis of the sum of the interior angles.

Referring to Figure 7, let us start from a triangle ABC (the blue one). First of all, let us consider the lines which identify the sides of the triangle as Euclidean semicircles, then let us create their central angles subtended by the sides of the triangle. We call them α, β and γ, with β between the other two angles. Secondly let us create the Euclidean triangle (the green one) with the same vertices of the triangle.

Referring to Figure 8, we start analysing the angle $\widehat{A}$. It is identified by the two tangents (the black Euclidean lines) at the semicircles passing through A, and it is divided in two parts by the side AC of the Euclidean triangle.

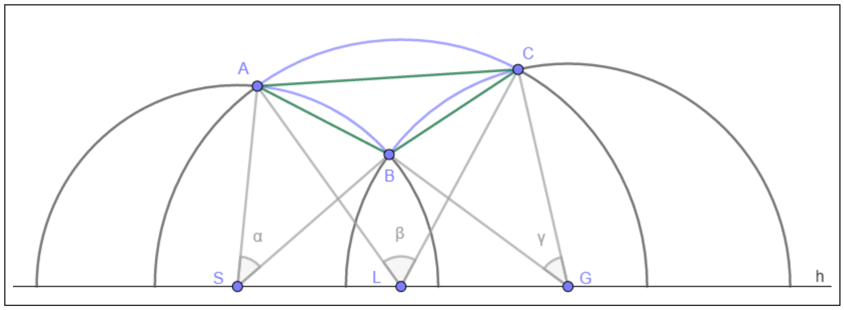

Figure 7

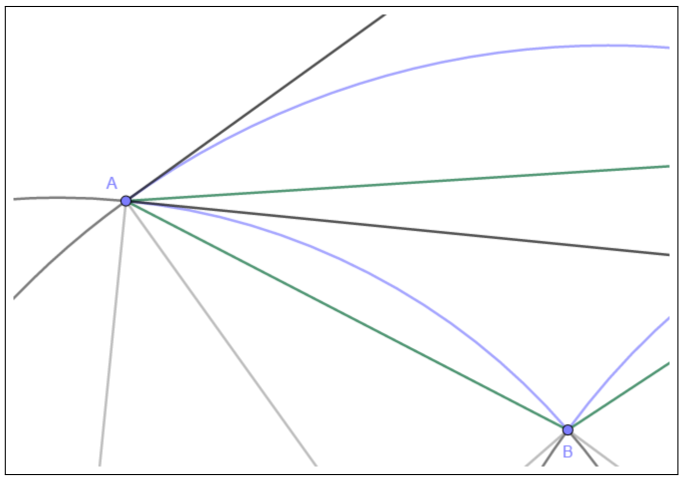

Figure 8

The angle identified by the upper black Euclidean line and the green side AC, is the angle at the circumference of the central angle β, so it measures $\frac{\beta}{2}$. The under part (angle identified by the green side AC and the lower Euclidean black tangent) is a part of the Euclidean angle $B\widehat{A}C$. In particular it measures $B\widehat{A}C - \frac{\alpha}{2}$, with $B\widehat{A}C$ the angle of the green triangle and $\frac{\alpha}{2}$ the angles identified by the green side AB and the lower black tangent, which is the angle at the circumference of the central angle α. Thus, the angle $\widehat{A}$ measures

$$\widehat{A} = B\widehat{A}C - \tfrac{\alpha}{2} + \tfrac{\beta}{2}.$$

The second angle we analyse is $\widehat{C}$. In a similar way the angle $\widehat{C}$ measures

$$\widehat{C} = B\widehat{C}A - \tfrac{\gamma}{2} + \tfrac{\beta}{2}.$$

We finally analyse the angle $\widehat{B}$. It is a part of the green Euclidean angle $A\widehat{B}C$. In particular the part on the left of $\widehat{B}$ (angle identified by the green side AB and the left Euclidean black tangent) is the angle at the circumference of the angle α, the part on the right of $\widehat{B}$ (angle identified by the green side BC and the right Euclidean black tangent) is the angle at the circumference of γ. Thus, the angle $\widehat{B}$ measures

$$\widehat{B} = A\widehat{B}C - \tfrac{\alpha}{2} - \tfrac{\gamma}{2}.$$

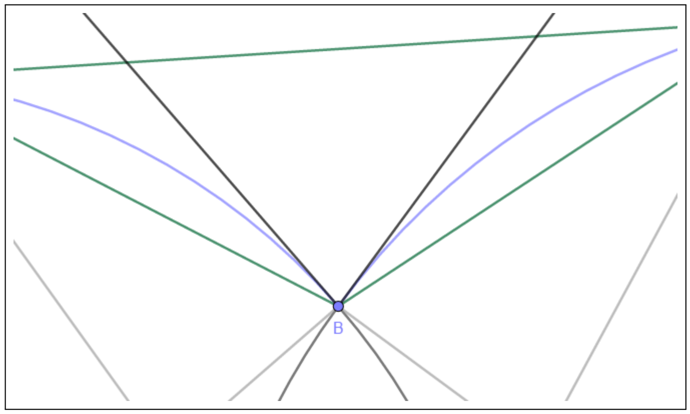

Figure 9

Now, summing the three angles $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, we find a general result for the sum of the interior angle of triangles, knowing the measures of the central angles. The value is $\pi - \alpha + \beta - \gamma$. This relation works only for this particular position of the vertex B with respect to the side AC.

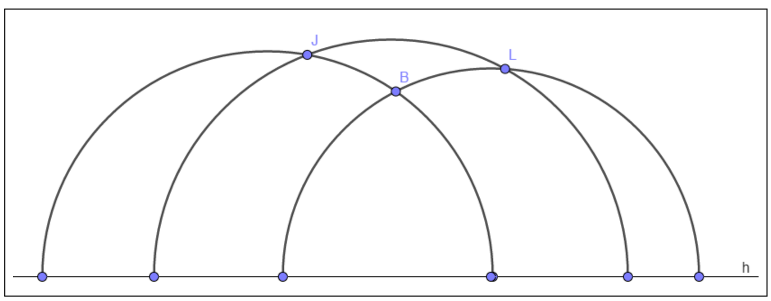

Figure 10

Instead, if B is above the segment AC, then the value is $\pi + \alpha - \beta + \gamma$.

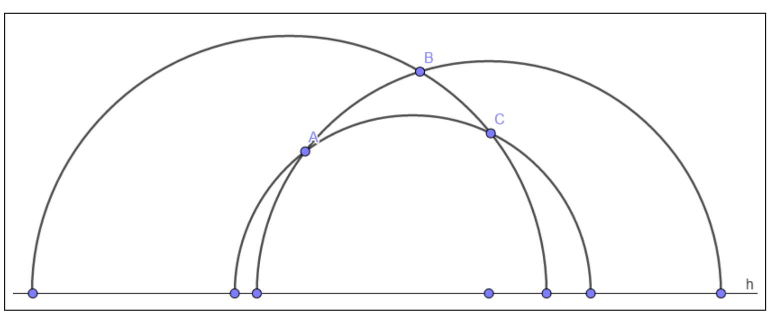

Figure 11

Hence, joining the two relations, we find the following:

Theorem. *Given a general triangle ABC identified by the intersections of three Euclidean semicircles, and called α, β and γ the three central angles of the Euclidean semicircles subtended by the sides of the triangles, with β between the other two angles, the sum of the interior angles of ABC is*

$$\pi \pm (-\alpha + \beta - \gamma) .$$

At this point we prove that:

Corollary. *The sum of the interior angles of triangle is less than π.*

We prove the statement starting with the first case, when the sum of the interior angles is equal to $\pi - \alpha + \beta - \gamma$. Proving $\pi - \alpha + \beta - \gamma < \pi$ is equivalent to show that $\alpha + \gamma > \beta$.

Referring to Figure 12, let us consider a triangle ABC and allow the existence of the half-plane below h. So let us consider the lines which identify the sides of the triangle as Euclidean circles and let us create the central angles of the three circles c, e and d. We call α the central angle of the circle c, β the central angle of e and γ the central angle of d. Then we call K the intersection between c and e, and Q the intersection between d and e (in the half-plane below h) .

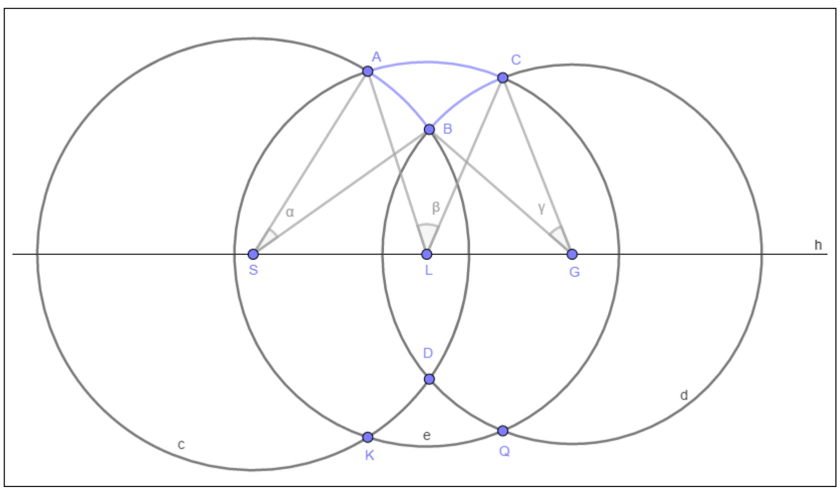

Figure 12

Let us consider the angle at the circumference starting from K and passing through AB. It is an angle at the circumference of both c (angle $A\widehat{K}B$) and e (angle $A\widehat{K}M$, with M the intersection with e found by the angle, which is subtended by AM). Let us consider the angle at the circumference starting from Q and passing through BC. As before, it is an angle at the circumference of both d (angle $C\widehat{Q}B$) and e (angle $C\widehat{Q}N$) and it finds an intersection with e, that we call N.

Then let us draw the central angle on e subtended by CN (angle $C\widehat{L}N$) and the one subtended by AM (angle $A\widehat{L}M$).

We know that $A\widehat{K}B = \frac{\alpha}{2}$ and that $A\widehat{K}B = A\widehat{K}M$, so $A\widehat{K}M = \frac{\alpha}{2}$. Since $A\widehat{K}M = \frac{1}{2}A\widehat{L}M$, $A\widehat{L}M = \alpha$. Similarly $C\widehat{Q}B = \frac{\gamma}{2}$ and $C\widehat{Q}B = C\widehat{Q}N$, so $C\widehat{Q}N = \frac{\gamma}{2}$. Since $C\widehat{Q}N = \frac{1}{2}C\widehat{L}N$, so $C\widehat{L}N = \gamma$. As you can see from the figure above $A\widehat{L}M + C\widehat{L}N > \beta$.

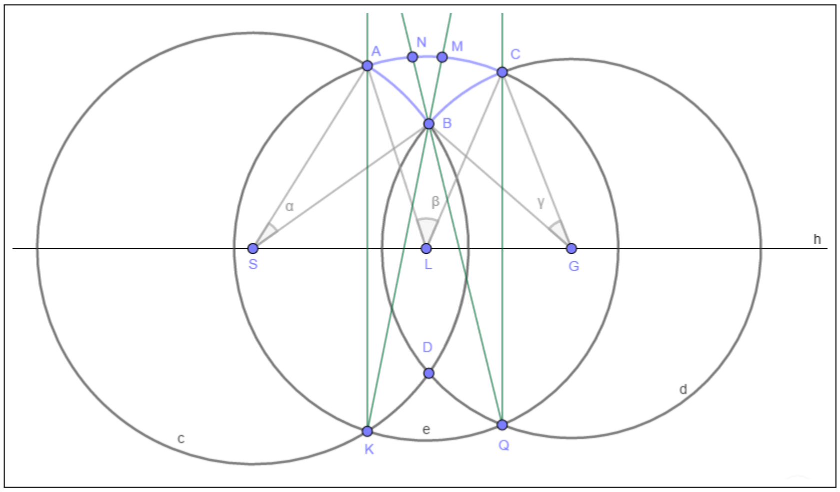

Figure 13

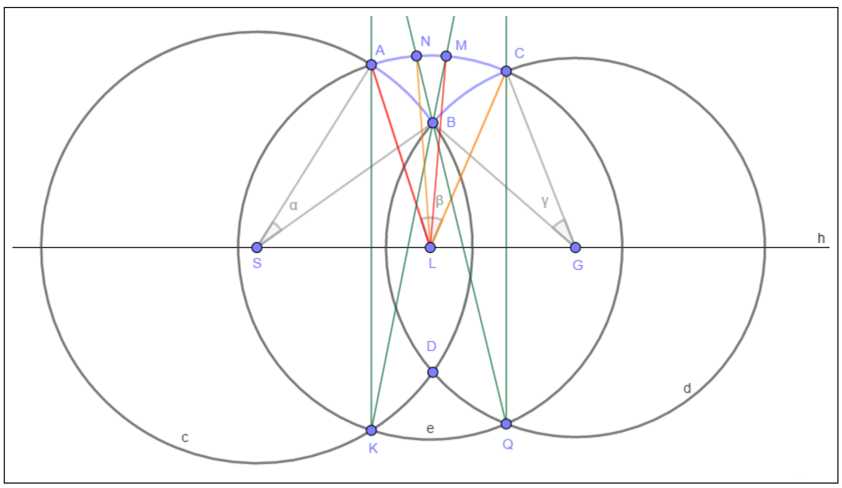

Figure 14

This is always true because, in the half-plane above h, $c \cap d < c \cap e$ and $c \cap d < d \cap e$ and, in the half-plane below h, $c \cap d > c \cap e$ and $c \cap d > d \cap e$. For this reason, the Euclidean line passing trough KB intersects e on M, which is on the right side of B, and the Euclidean line passing trough QB intersects e on N, which is on the left side of B.

Thus, since $A\hat{L}M + C\hat{L}N > \beta, \quad \alpha + \gamma > \beta$ and as a consequence the sum of the interior angles of this type of triangles is less than π.

Let us now consider the second case, in which we have to prove that $\alpha + \gamma < \beta$.

The second case lead us to the same conclusion of the first one. We consider the same plane of before, but with a different triangle ABC which have the vertex in the middle above its opposite side.

Let us do the same procedure we have just done. This time $A\hat{L}M + C\hat{L}N < A\hat{L}C$. This is always true because, in the half-plane above h, $c \cap d > c \cap e$ and $c \cap d > d \cap e$ and, in the half-plane below h, $c \cap d < c \cap e$ and $c \cap d < d \cap e$. For this reason, the Euclidean line passing

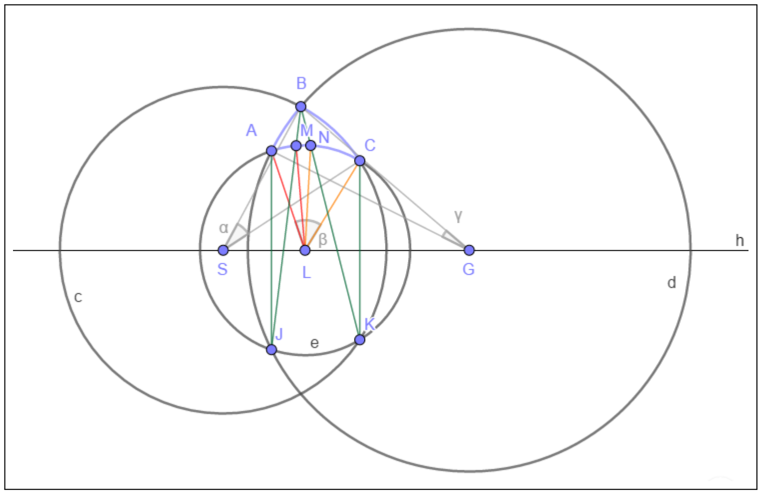

Figure 15

through KB intersects e on M, which is on the left side of B, and the Euclidean line passing through QB intersects e on N, which is on the right side of B.

Thus, since $A\widehat{L}M + C\widehat{L}N < \beta, \quad \alpha + \gamma < \beta$ and as a consequence the sum of interior angles of this type of triangles is $< \pi$.

In conclusion, we proof that the sum of interior angles of all triangles is always less than π.

4.3 Sum of interior angles of all polygons

The relation we have just proved can be generalized to all polygons. The generalization works exactly as in the Euclidean case:

Corollary. *The sum of interior angles of all the types of polygons is less than $\pi \cdot (n - 2)$, with n the number of the sides of the polygon.*

5 Triangles

5.1 Definition

A triangle is polygon delimited by three sides.

5.2 Criteria of congruence

Two triangles are congruent if it is possible to overlap them with a rigid movement so that they coincide point by point (4). Some necessary and sufficient conditions for a pair of triangles to be congruent are:

- SAS Criteria: Two sides and the angle between them in a triangle are congruent to those in the other triangle, respectively.
- ASA Criteria: Two angles and the side included (the side that they have in common) in a triangle are congruent, respectively, to those in the other triangle.

- SSS Criteria: Each side in a triangle is congruent, respectively, to the corresponding one in the other triangle.

We assume the SAS criteria as a postulate and, as a consequence, the proves of the other two criteria are the same as the Euclidean ones.

5.3 Notable points

It is possible to prove that even in this non-Euclidean geometry, the four notable points of the triangles exist.

1. *Incenter*: it is the intersection point of all the bisectors of a triangle.

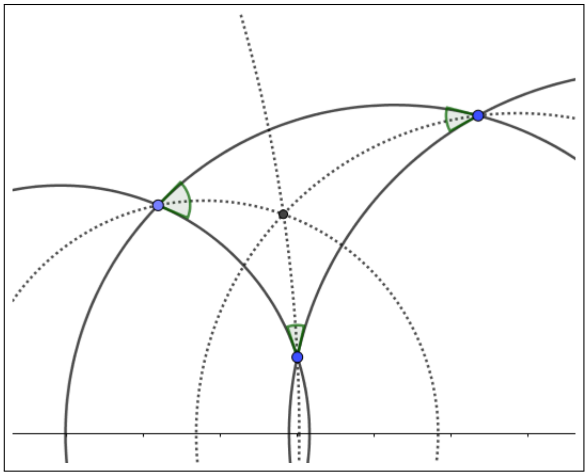

Figure 16: Incenter

2. *Circumcenter*: it is the intersection point of the three perpendicular bisectors of a triangle.

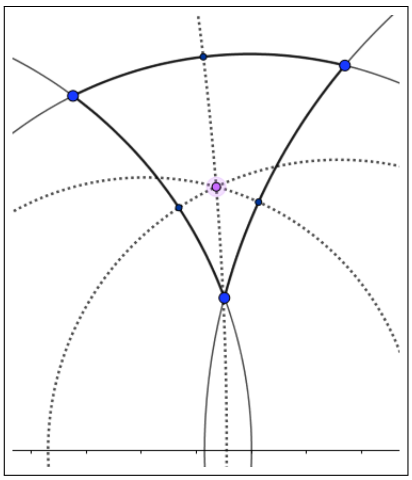

Figure 17: Circumcenter

3. *Centroid*: it is the intersection point of the medians of a triangle.

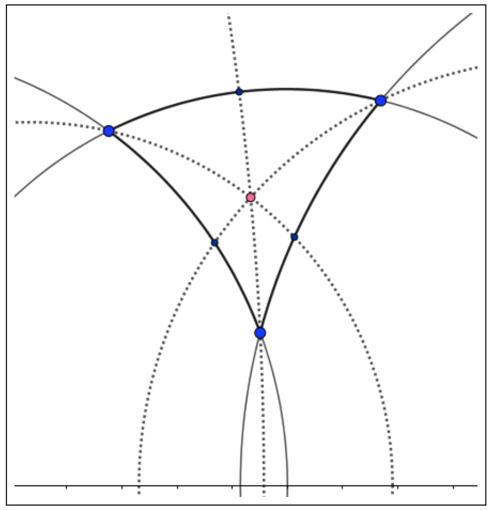

Figure 18: Centroid

4. *Orthocenter*: it is the intersection point of the three altitudes of a triangle.

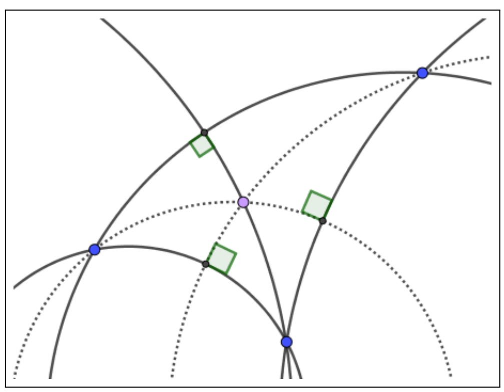

Figure 19: Orthocenter

The proof of existence of both incenter and circumcenter is the same as the Euclidean one (5). On the other hand, we failed to prove the other ones because in Euclidean geometry they exploit the Thalete Theorem, which rests on the fifth postulate, denied from our problem.

5.4 Classification with respect to sides

The triangles can be classified according to their sides

- *Scalene triangle*: it has three non-congruent sides. It is noticeable that it has three different angles, as in the Euclidean geometry.

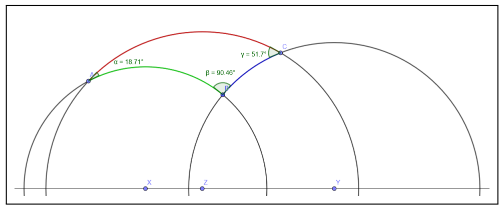

Figure 20: Scalene Triangle

- *Isosceles triangle*: it has two congruent sides. It is noticeable that it has congruent base angles as in the Euclidean geometry. The proof of the Isosceles angle theorem is the same as the Euclidean one.

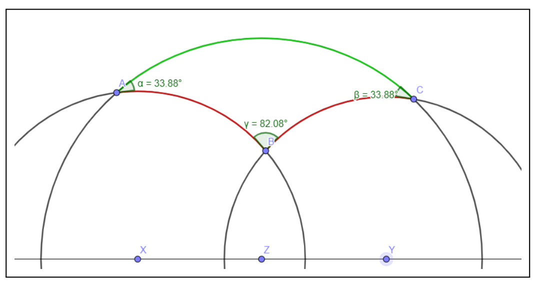

Figure 21: Isosceles tTriangle

- *Equilateral triangle*: it has three congruent sides. It is noticeable that it has three congruent angles as in the Euclidean geometry. The angles measure is not 60°, but it depends on the side measure.

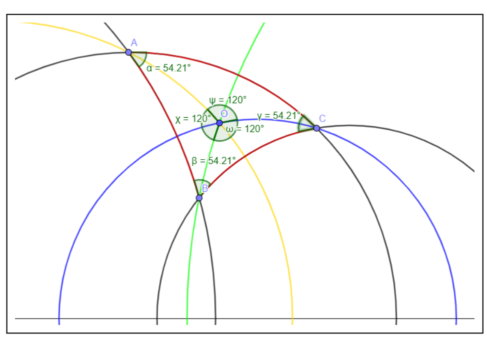

Figure 22: Equilateral Triangle

5.5 Classification as regard its angles

The triangles can be classified according to their angles, here measured in degrees.

- *Acute Triangle*: it is a triangle in which all the interior angles measure less than 90°.

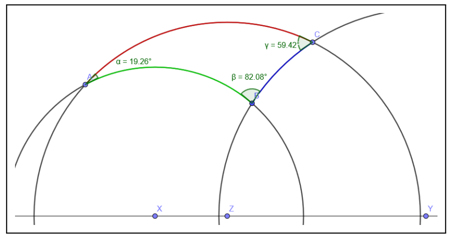

Figure 23: Acute triangle

- *Obtuse triangle*: it is a triangle in which one interior angle measures more than 90°.

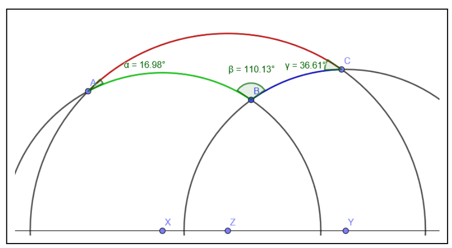

Figure 24: Obtuse Triangle

- *Right-angled triangle*: it is a triangle with one of its interior angles measuring 90°. As regards the Pythagorean theorem and the Euclidean theorem, we have not been able to prove them, not having a definition of length, therefore of area.

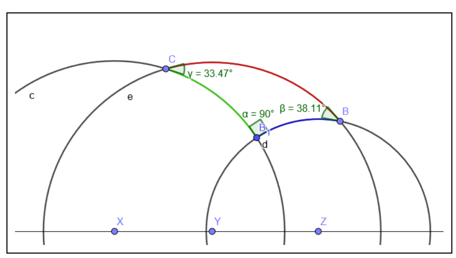

Figure 25: Right-angled Triangle

6 Quadrilaterals

6.1 Definition

A quadrilateral is a polygon delimited by four sides. Referring to [4.3], we can notice that the sum of interior angles of all quadrilaterals is always less than 360°.

6.2 Parallelogram

It is defined as a quadrilateral with opposite angles and sides (6). Indeed, it is not possible to define it in the exact same way of the Euclidean one due to the denial of the fifth axiom. We can notice that all the properties of the Euclidean parallelogram are verified.

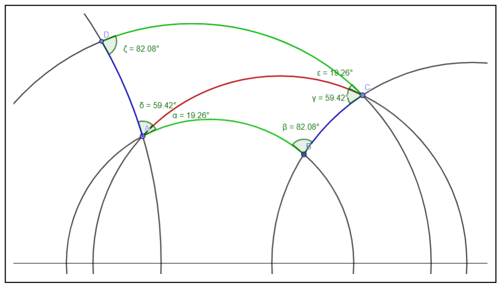

Figure 26: Parallelogram

6.3 Rhombus

It has four congruent sides and the opposite angles are congruent. All the properties of the Euclidean rhombus are verified.

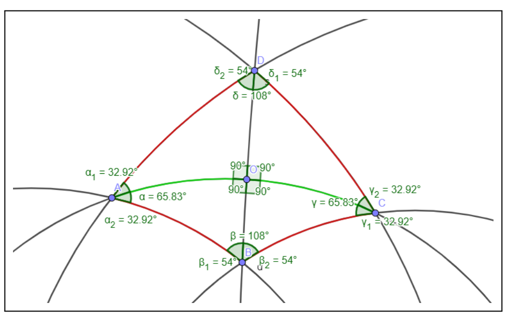

Figure 27: Rhombus

6.4 Rectangle

It is a parallelogram with all congruent angles. Their measure is not 90°, but it depends on the side measure. All the properties of the Euclidean rectangle are verified.

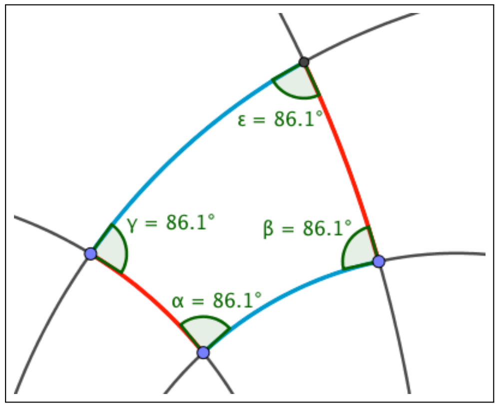

Figure 28: Rectangle

6.5 Square

It is a rectangle with four congruent sides. It has four congruent angles as in the Euclidean geometry. Their measure is not 90°, but it depends on the side measures. All the properties of the Euclidean square are verified.

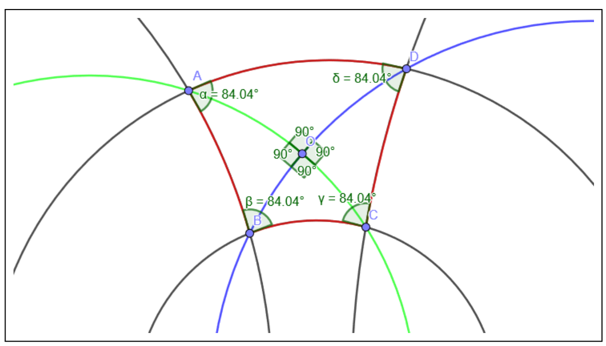

Figure 29: Square

7 Regular polygons

7.1 Definition

A regular polygon is a polygon delimited by congruent sides (7). Here we analyse regular polygons with at least five sides.

7.2 Pentagon

It has five congruent sides and angles. All the properties of the Euclidean pentagon are verified. The sum of the interior angles is less than 540°.

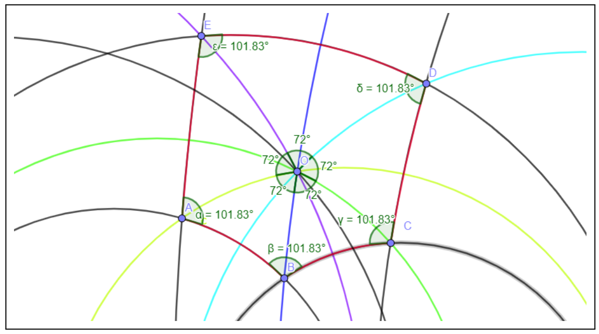

Figure 30: Pentagon

7.3 Hexagon

It has six congruent sides and angles. All the properties of the Euclidean hexagon are verified. The sum of the interior angles is less than 720°.

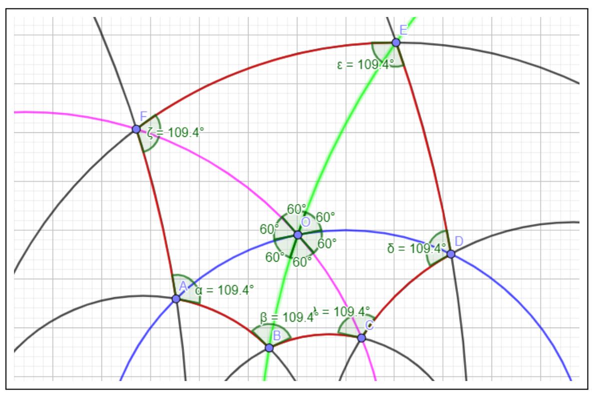

Figure 31: Hexagon

7.4 Octagon

It has eight congruent sides and angles. All the properties of the Euclidean octagon are verified. The sum of the angles is less than 1080°.

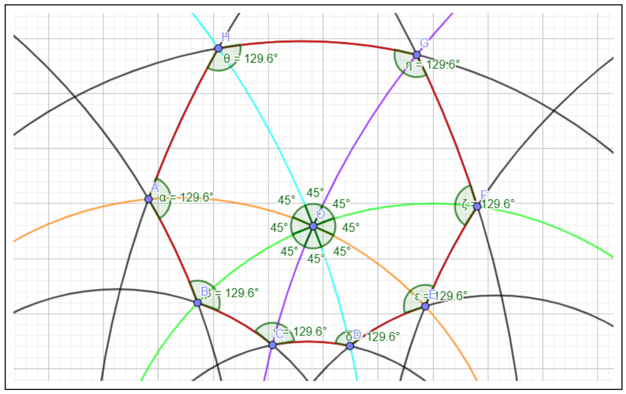

Figure 32: Octagon

8 Circle

A circle centered in C is the locus of the points P such that the segments PC are congruent when P varies. Visually, it does not appear perfectly circular, but more vertically flattened (8).

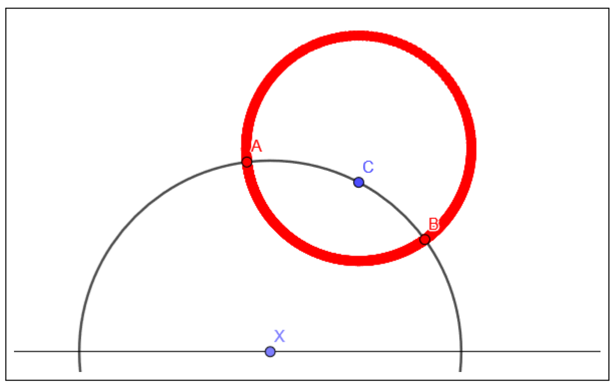

Figure 33: Circle

9 Constructions

In this section we explain how to construct some polygons.

9.1 Triangles

9.1.1 Scalene Triangle

Let us create three lines intersecting one another. The triangle is the polygon formed by the three segments generated by the intersections. You can check whether the sides are congruent with the rigid motion.

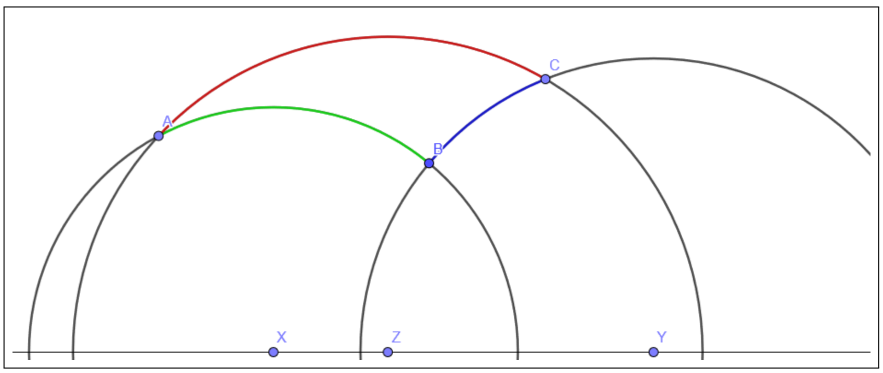

Figure 34: Scalene Triangle

9.1.2 Isosceles Triangle

Let us create a segment on a line. Using the rigid motion let us create a congruent segment with a shared endpoint.

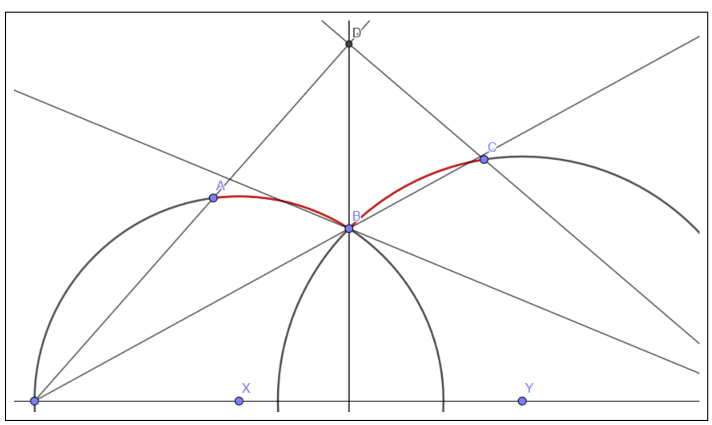

Figure 35

Using the Euclidean perpendicular bisector of the two non shared endpoints we find the centre of the third side's line.

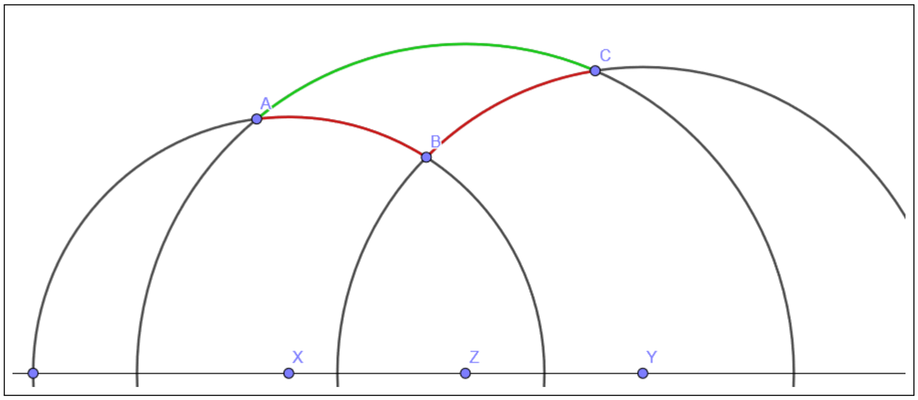

Figure 36: Isosceles Triangle

9.1.3 Equilateral Triangle

Let us create a line and the point O being on this line. Let us create the Euclidean tangent line to the line passing through O and let us divide the turn angle in three angles, each measuring 120°. The Euclidean half-lines perpendicular to the Euclidean lines that define the angles we just described can be used to find the other lines where the vertices of the equilateral triangle are going to be.

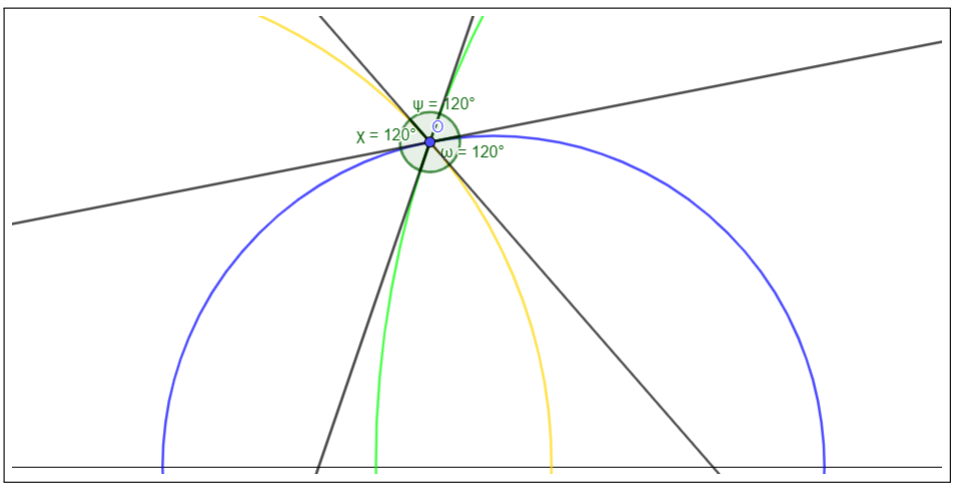

Figure 37

You now have to choose where to put the vertex A of your triangle on the first non-Euclidean line and then, using the rigid motion, finding the other vertices B and C.

After that, as you do while creating an isosceles triangle you use the perpendicular bisectors of the Euclidean segments AB, BC and CA to find the centre of the non-Euclidean lines that contain the sides of triangle found joining the vertices.

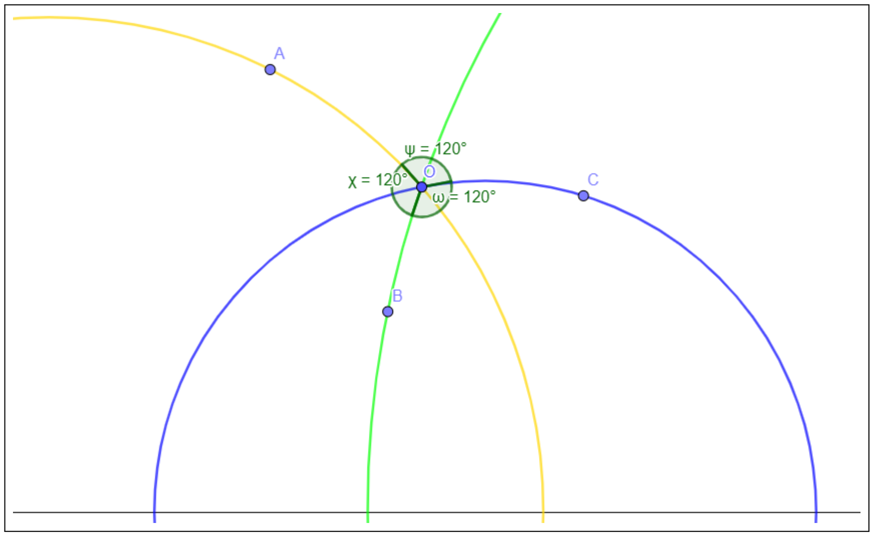

Figure 38

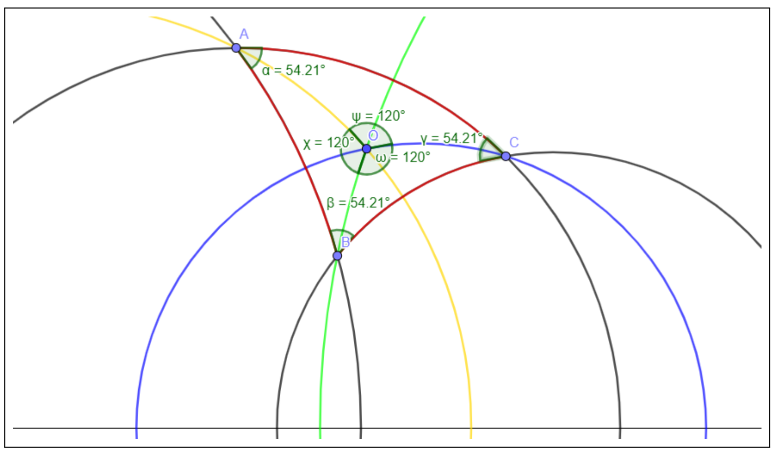

Figure 39: Equilateral Triangle

9.1.4 Acute Triangle

Construct a scalene triangle while making sure no angle is greater than or equal to 90° using the Euclidean tangents of the non-Euclidean lines passing through the vertex.

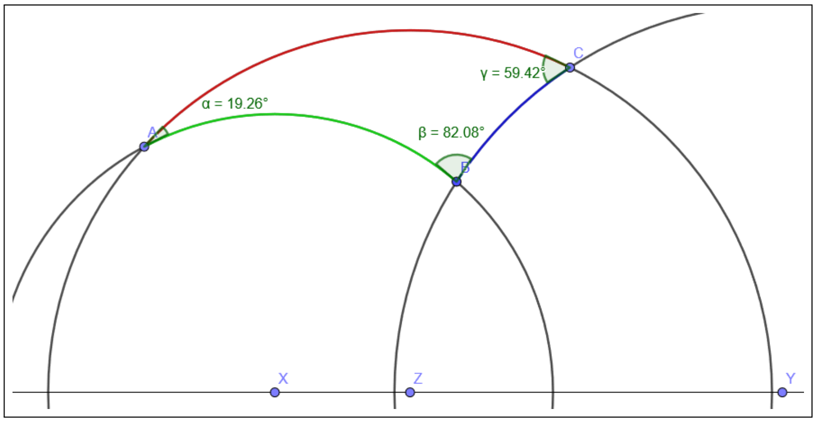

Figure 40

You can also find an isosceles acute triangle by making sure the angle between the two congruent sides is less than or equal to 90°.

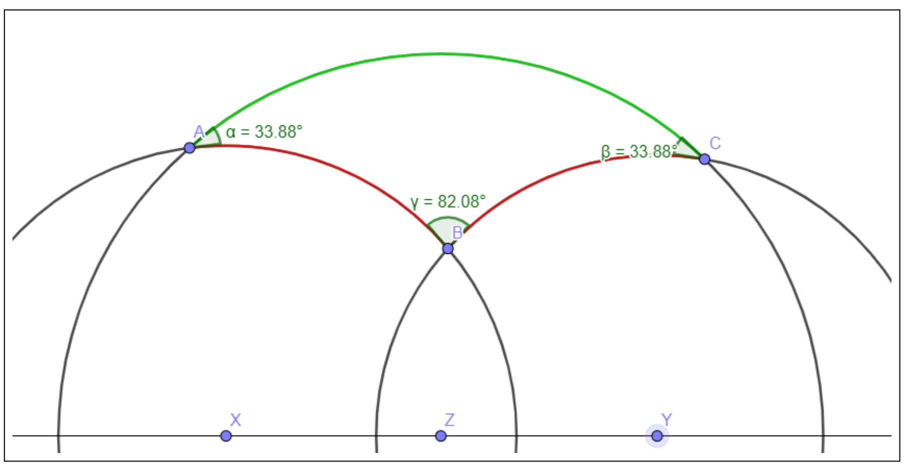

Figure 41

An equilateral triangle is always acute in this non-Euclidean geometry

9.1.5 Obtuse Triangle

Construct a scalene triangle while, using the Euclidean tangents of the non-Euclidean lines passing through the vertex, making sure that there is an angle greater than or equal to 90°.

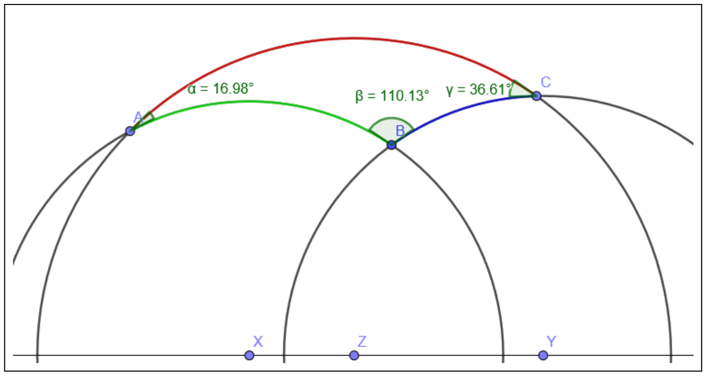

Figure 42

You can also find an isosceles obtuse triangle by making sure that the angle between the two congruent sides is greater than or equal to 90°.

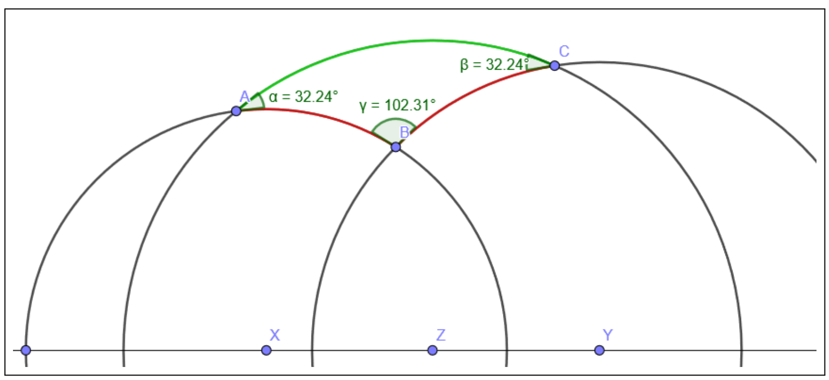

Figure 43

9.2 Quadrilaterals

9.2.1 Parallelogram

Let there be a triangle ABC; create an angle adjacent to $C\widehat{A}B$ congruent to $B\widehat{C}A$ and an angle adjacent to $B\widehat{C}A$ congruent to $C\widehat{A}B$; using the Euclidean perpendicular lines to the half-lines that delimit the angles that we just created, we can find the centre of the non-Euclidean lines where the other two sides of the Parallelogram are located; the intersection of these two non-Euclidean lines is the fourth vertex and the quadrilateral is made of the four congruent sides of the two triangles.

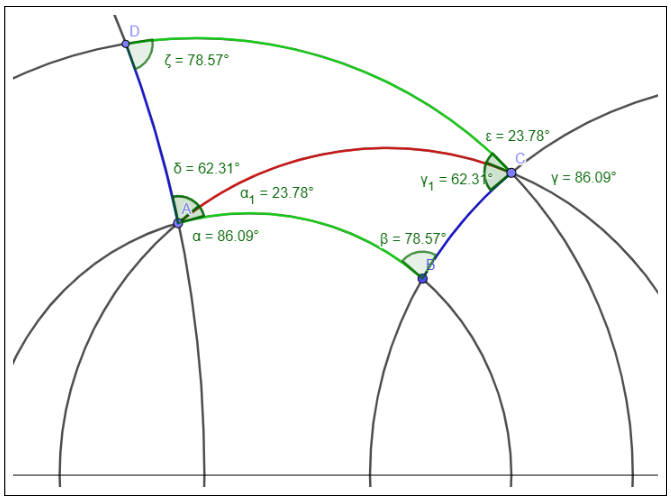

Figure 44: Parallelogram

9.2.2 Rhombus

Let there be an isosceles triangle ABC; create a congruent angle adjacent to each angle at the base of the triangle; using the Euclidean perpendicular lines to the new half-lines that delimit the angles that were just created, we can find the centre of the non-Euclidean lines where the other two sides of the Rhombus are located; the intersection of these two non-Euclidean lines is the fourth vertex and the quadrilateral is made of the four congruent sides of the two isosceles triangles.

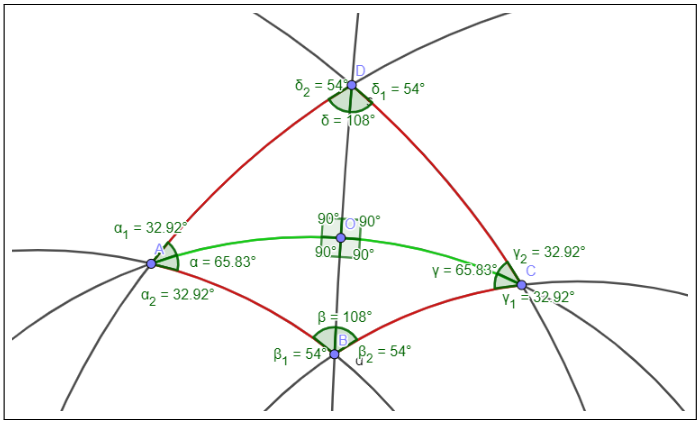

Figure 45: Rhombus

9.2.3 Square

The construction just like all the other regular polygons is similar to the equilateral triangle one, but dividing the turn angle in four 90° angles instead.

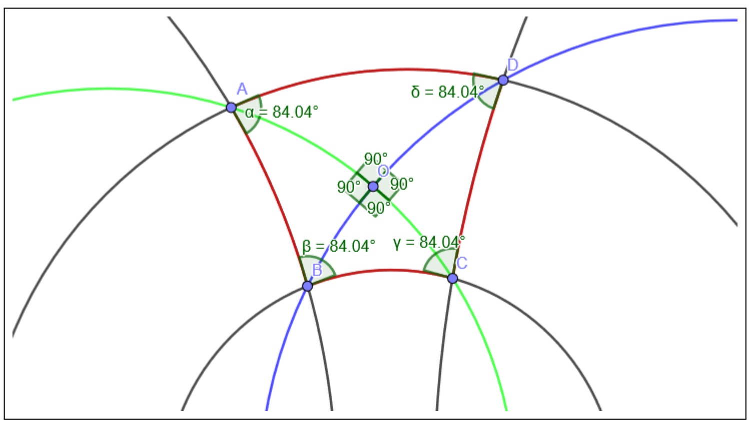

Figure 46: Square

9.3 Regular Polygons

9.3.1 Pentagon

Regular polygon construction with five 72° angles.

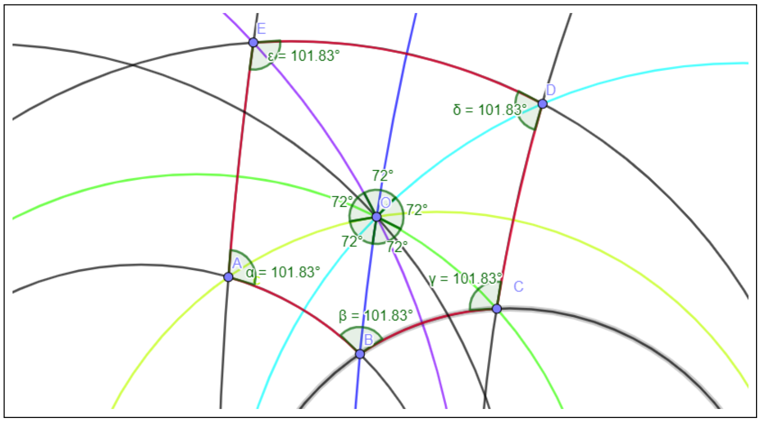

Figure 47: Regular pentagon

9.3.2 Hexagon

Regular polygon construction with six 60° angles.

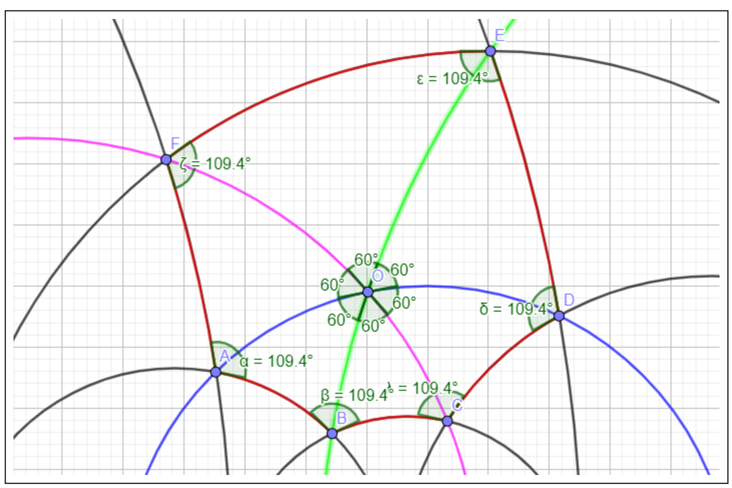

Figure 48: Regular hexagon

9.3.3 Octagon

Regular polygon construction with eight 45° angles.

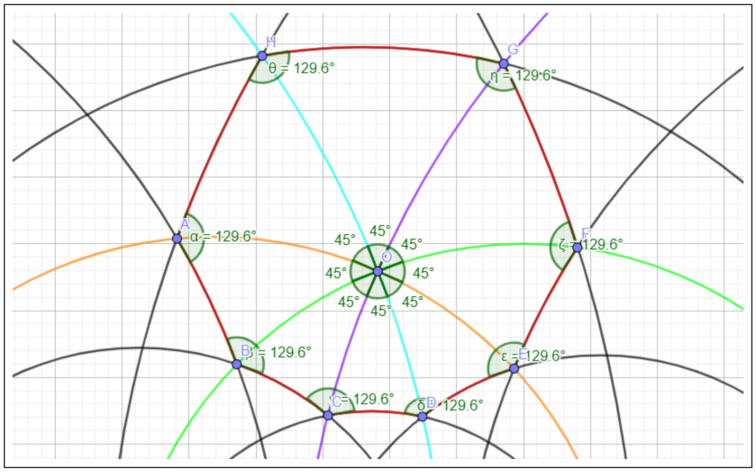

Figure 49: Regular octagon

9.4 Circle

Given a point C as the centre of the circle, choose a radius and a point D of the circle on the degenerate line throuh C. Then, using the rigid motion, find the points of the circle on any non-Euclidean line through C. Moving the centre of the non-Euclidean line will compose the circle.

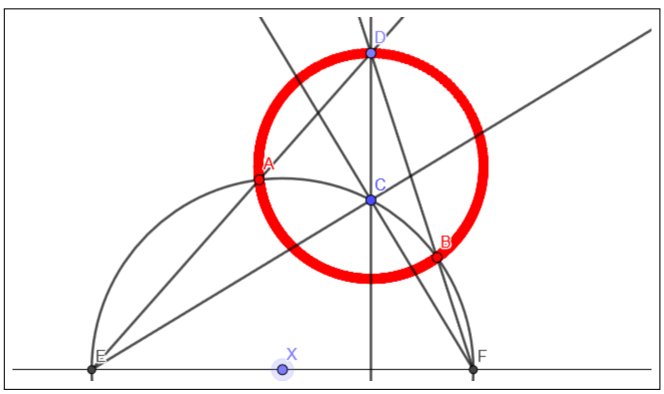

Figure 50

References

[1] D. Hilbert, *The Foundations of Geometry*, 2nd ed., Chicago: Open Court, 1980 (1899).

[2] F. Rothe, *Hyperbolic Geometry and the Pseudo-sphere*, math2.uncc.edu (2006).

Editing Notes

(1) *Warning from editors.* The first part of this work, up to section 4, presents a thorough study, consistent with what is desired for a MeJ article. However the second part, from section 5 onwards, consists mostly in definitions, constructions or results given without proofs and, despite their interest as well as the care given to the numerous figures, this latter part is very incomplete.

(2) The list of axioms given here is not complete. In particular, the axioms concerning angles are missing (axioms IV-4 to IV-6 in Hilbert's book [1]). We also note that Hilbert introduces the notion of congruence of segments and angles, but does not refer to rigid motions his axioms.

(3) Note that in this section, the authors present only the action of rigid motions on lines and segments, but no information is given on their action on the angles.
Besides, for ease of reading, the reader may begin with paragraphs 3.3 (from a line to a degenerate right) and 3.4 (from a degenerate line to a different degenerate line). In other cases, an auxiliary degenerate line is introduced, and we can proceed by transitivity.

(4) As said in Note 1, from here the results are given without real proofs.
Moreover, we do not have here an effective definition of the congruence of triangles, since we do not have a characterization of the rigid movements except for their action on a line. Also, orientation may be changed and 'rigid movement' should be replaced by 'isometry'.

(5) This assertion is not satisfactory, since Euclidean proofs rely on the notion of distance which is not defined here.
We also remark that the result on the orthocenter holds when two altitudes intersect each other but the three altitudes may be parallel.

(6) A parallelogram is a quadrilateral with *congruent* opposite angles and sides.

(7) A regular polygon is a polygon with congruent sides *and congruent angles*.

(8) Actually, hyperbolic circles are true Euclidean circles. The reader may prove it by looking at Figure 50 (*Hint:* show that the intersection of the Euclidean line AF with the degenerate line through C is the second point of the circle on this degenerate line).