Game of differences

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The question

There are 4 natural numbers a, b, c, d written on a row. The differences |a-b|, |b-c|, |c-d|, |d-a| are then written on the next row. The process is continued:

7	**3**	**9**	**2**
4	**6**	**7**	**5**
2	**1**	**2**	**1**
1	**1**	**1**	**1**
0	**0**	**0**	**0**

It can be observed that the null row 0, 0, 0, 0 has been obtained. Is this a coincidence or is the null row obtained for any natural numbers a, b, c, d?

Can it be obtained from any 4 real numbers?

What happens if the rows don't have 4 numbers, but 3, 5, 6... ?

Solution :

What happens if we have:

1. **4 natural numbers**
2. **5, 6, 7,.. natural numbers**
3. **4 rational numbers**
4. **4 real numbers**

1.

1.1 Solution 1:

We replace the numbers with their remainders modulo 2. There are 2^4=16 ways to distribute the remainders 0 and 1 on the first row. All these posibilities are shown in the following table:

1	0	1	1	1	1	0	1	1	1	1	0	0	1	1	1
or				or				or				or			
0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0
1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For example, if we start with 4 numbers that have the remainders (1,0,1,1), in this order, after 4 steps, the numbers obtained will have the remainders (0,0,0,0), as shown in the table. But if we start with 4 positive integers with remainders (0,1,0,1) modulo 2, after only 2 steps the numbers will be divisible by 2.

So, to conclude, after at most 4 steps, the obtained numbers will be divisible by 2. The remainders at the division by 2^2=4 when we divide the last numbers obtained are 0 or 2. If we make identical tables with the one made before, but instead of using the pair (0,1) we are using the pair (0,2) of remainders, we notice that, after at most another 4 steps, all the numbers obtained will be divisible by 4.We continue the process. Inductively, **after at most 4k steps, all the numbers on the last row will be divisible by 2^k.**

There is for sure a natural number m so that all the four numbers on the first row are smaller than 2^m. As we deduced before, after 4(m+1) steps, all the numbers on the last row will be divisible, for sure, by 2^{m+1}. Let $a_1 = a, b_1 = b, c_1 = c, d_1 = d$ and $a_n, b_n, c_n, d_n \in \mathbb{N}$, $a_n = |a_{n-1} - b_{n-1}|$, $b_n = |b_{n-1} - c_{n-1}|$, $c_n = |c_{n-1} - d_{n-1}|$, $d_n = |d_{n-1} - a_{n-1}|$.

Let $x_n = \max\{a_n, b_n, c_n, d_n\}$.

Taking into consideration the way sequences are defined and using that $|a-b| \le \max\{a,b\}$ for any positive integers a,b , we obtain that x_n is decreasing. So after 4(m+1) steps, the numbers obtained will be less than 2^m. The only posibility for them to be divisible by 2^{m+1} is if they are equal to 0.

In conclusion, after a certain number of steps, we will reach the null row (0, 0, 0, 0), regardless of the four positive integers that we choose.

2.**What happens if on the first line there are not four numbers, but three, five , six or seven?**

We will show you that it is possible that, for certain configurations of the first line, to not achieve the zero configuration at the end. But, as said before, what is going to happen if we choose to have a different amount of numbers on the first line?(1) We are going to prove that for some ways of choosing the configurations on the first line, we will enter in a cycle. It follows that in these cases there will never be a line of zeroes.

Example of a cycle in the case of 3 numbers in a row:

0	1	1
1	0	1
1	1	0
0	1	1

- cycle of length 3-

Example of a cycle in the case of 5 numbers in a row:

0	0	0	1	1
0	0	1	0	1
0	1	1	1	1
		or		
1	0	0	0	0
1	0	0	0	1
		or		
0	1	1	1	0
1	0	0	1	0
		or		
0	1	1	0	1
1	0	1	1	1
		or		
0	1	0	0	0
1	1	0	0	0
		or		
0	0	1	1	1
0	1	0	0	1
		or		
1	0	1	1	0
0	1	1	0	0
		or		
1	0	0	1	1
1	0	1	0	0
		or		
0	1	0	1	1
1	1	1	0	1
		or		
0	0	0	1	0
		or		
0	0	1	1	0
0	1	0	1	0
		or		
1	0	1	0	1
1	1	1	1	0
0	0	0	1	1

- cycle of length 14-

Example of a cycle in the case of 6 numbers in a row:

0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	1
1	1	0	0	1	1
0	1	0	1	0	0
1	1	1	1	0	0
0	0	0	0	0	1

- cycle of length 6

Example of a cycle in the case of 7 numbers in a row:

0	0	0	0	0	1	1
0	0	0	0	1	0	1
0	0	0	1	1	1	1
0	0	1	0	0	0	1
0	1	1	0	0	1	1
1	0	1	0	1	0	1
1	1	1	1	1	1	0
0	0	0	0	0	1	1

- cycle of length 7-

If on the first line there are eight numbers, they will become 0 after a number of steps as well.

To prove this, we have written a C++ program that demonstrates that, no matter the order of the zeros and ones on the first line of the table, they will become zeros eventually. Using the idea from the first solution, we replace the numbers with their remainders mod 2. The program we wrote shows that after 8 steps one obtains a line of zeroes, so the numbers will be divisible by 2. So, after 8k steps the numbers are all divisible by 2^k . An argument similar to the case of 4 numbers in a row shows that a line of zeroes is obtained eventually. It results that the numbers will become 0 after a maximum of 8k steps.

The program generates the numbers between 0 and 255 and converts them to binary to obtain all the 8 digit numbers that represent the possible combinations of 0 and 1. Then, it implements the algorithm described in the rubric, until it reaches a null row or until it becomes a cycle. To be able to check at the end if all the combinations check the condition, we have taken a variable 'ok', which increases each time a combination checks the condition.

Result: The condition is true for all combinations.

3. What happens if on the first line there are four rational numbers?

First of all, we demonstrate that if the first line is *L = (a, b, c, d)* or *tL = (ta, tb, tc, td)*, the algorithm leads (or not) to zeros after the same number of steps. If we start with four rational numbers on the first line of the table, let *t* be a common multiple of the four denominators. The *tL* line will contain four natural numbers and, as shown above, at one point will be a line which consists of just zeros. It follows that the initial line will also lead to a zero line.(2)

$\frac{a_1}{b_1}$	$\frac{a_2}{b_2}$	$\frac{a_3}{b_3}$	$\frac{a_4}{b_4}$
d_1	d_2	d_3	d_4
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
x_1	**x_2**	**x_3**	**x_4**

Let us consider $t = b_1 b_2 b_3 b_4$.

$a_1 \times b_2 \times b_3 \times b_4$	$a_2 \times b_1 \times b_3 \times b_4$	$a_3 \times b_1 \times b_2 \times b4$	$a_4 \times b_1 \times b_2 \times b_3$
$t \times d_1$	$t \times d_2$	$t \times d_3$	$t \times d_4$
$t \times x_1$	$t \times x_2$	$t \times x_3$	$t \times x_4$

4. Is our result true if we choose some random four real numbers for the first line?

The answer is ***NO***! We can choose combinations of irrational numbers so such that a cycle appears. Let us consider the quadruple $M = (1, t, t^2, t^3)$ for the first line. By performing the operation, we will obtain on the second line $(t-1, t(t-1), t^2(t-1), (t^2+t+1)(t-1))$. Consider the polynomial $P(t) = t^3 - t^2 - t - 1$. As $P(1) = -2 < 0$ and $P(2) = 1 > 0$, it results that P has a solution $t_0 \in \left]1,2\right[$.
Considering the fact that P has no rational solutions (the only posibilities would have been 1 and −1, which don’t verify the equation)(3) , $\boldsymbol{t_0} \in \boldsymbol{R} \setminus \boldsymbol{Q}$.

For this value of t_0, it results that the second line is equal with $(t_0 - 1)M$. Inductively, the (n+1)th line of the table will be equal with $(t_0 - 1)^{n-1} M$, for any natural number *n*.

Line	Numbers				
1	**M**	**1**	t_0	${t_0}^2$	${t_0}^3$
2	**(t_0-1)M**	$t_0 - 1$	$t_0(t_0 - 1)$	${t_0}^2(t_0 - 1)$	$({t_0}^2 + t_0 + 1)(t - 1)$
.	.	.	.	.	.
.	.	.	.	.	.
n+1	$(t_0 - 1)^{n-1}M$	$1(t_0 - 1)^{n-1}$	$t_0(t_0 - 1)^{n-1}$	${t_0}^2(t_0 - 1)^{n-1}$	${t_0}^3(t_0 - 1)^{n-1}$

The result is that we will not get to have a line *n* made up only of zeros.

Notes d'édition

(1) La présentation du cycle de longueur 5 n'est pas très claire : pourquoi introduire une bifurcation à la troisième étape ? Il faudrait le présenter comme le cas de 6 nombres ou 7 nombres… De plus il y a une erreur à 8ieme étape

0	1	0	0	1

donne

1	1	0	1	1

et non pas

0	1	1	0	0

(une étape a été sautée apparemment …)

le tableau (d'une ligne) qui débute la page 4 devrait être remplacé par

0	0	0	1	0	1

(2) Ces deux tableaux n'apportent pas grand-chose. En un peu plus le texte, en disant par exemple que si on multiplie la première ligne par t alors l'algorithme produit des lignes qui sont les multiples par t des lignes initiales, cela permet de mieux comprendre

(3) Il n'est pas évident que P n'admette pas de solutions rationnelles, il faudrait expliquer ce point…

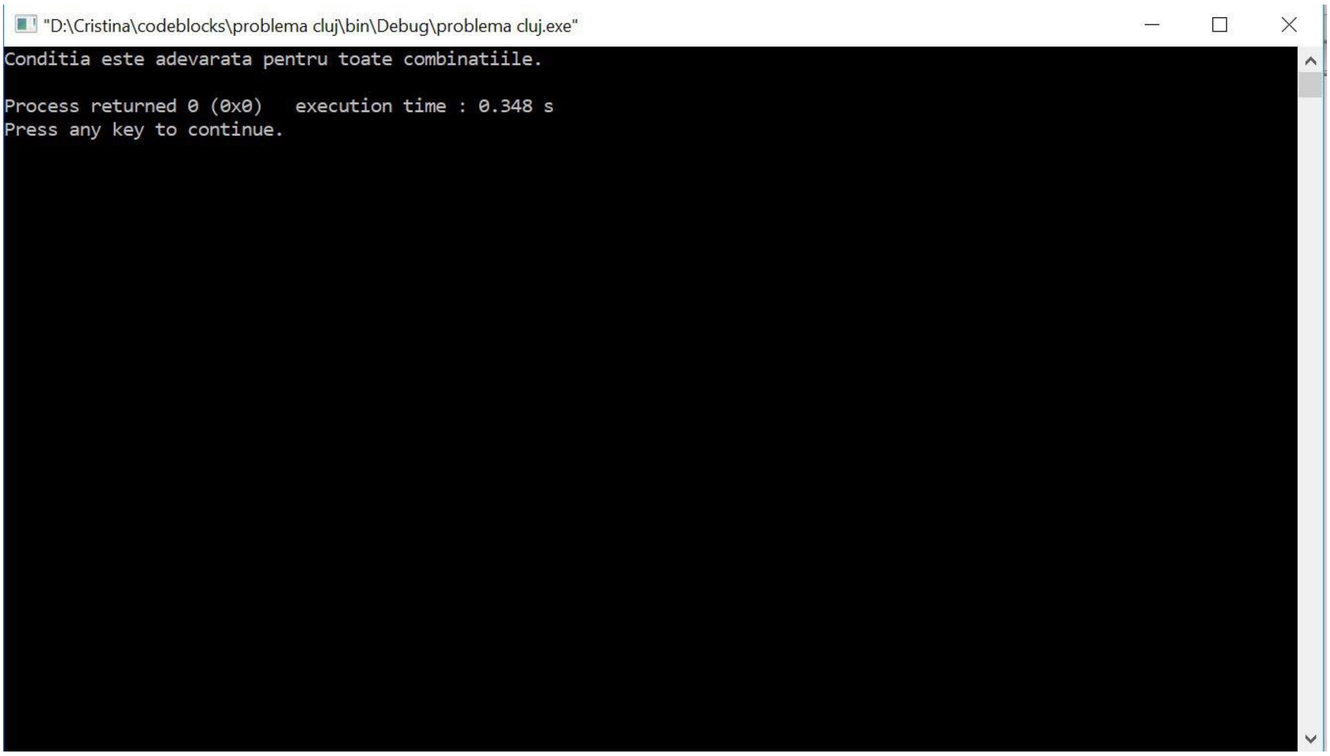
Annex 1

main.cpp - Code::Blocks 16.01

```cpp
#include <iostream>
#include <vector>

using namespace std;

vector <int> combinatii_8,copie_init;

int main()
{
    int ok=0;
    int baza2,f,cif,nr_cif,nr_0,primul,ci,nrpasi;
    for (int i=0;i<256;i++)
    {
        baza2=0;
        nr_0=0;
        nr_cif=0;
        f=1;
        ci=i;
        while (ci!=0)
        {
            cif=ci%2;
            baza2=baza2+cif*f;
            f=f*10;
            ci=ci/2;
            nr_cif++;
        }
        for (int j=1;j<=8-nr_cif;j++)
            combinatii_8.push_back(0);

        while (baza2!=0)
        {
            cif=baza2%10;
            combinatii_8.push_back(cif);
            baza2=baza2/10;
        }
        copie_init=combinatii_8;
        do
        {
            primul=combinatii_8[0];
            nr_0=0;
            for (int j=0;j<7;j++)
            {
                if (combinatii_8[j]>combinatii_8[j+1])
                    combinatii_8[j]=combinatii_8[j]-combinatii_8[j+1];
                else
                    combinatii_8[j]=combinatii_8[j+1]-combinatii_8[j];
            }
            if (combinatii_8[7]>primul)
                combinatii_8[7]=combinatii_8[7]-primul;
            else
                combinatii_8[7]=primul-combinatii_8[7];
            for (int j=0;j<8;j++)
                if (combinatii_8[j]==0)
                    nr_0++;
            nrpasi++;

        } while(nr_0<8 && combinatii_8!=copie_init);
        if (nr_0==8)
            ok++;
        else
            i=256;
    }
    if (ok==256)
        cout<<"Conditia este adevarata pentru toate combinatiile."<<endl;
    else
    {
        cout<<"Conditia nu este adevarata pentru toate combinatiile:"<<' ';
        for (int i=0;i<8;i++)
            cout<<combinatii_8[i]<<' ';

    }
    cout<<nrpasi;

    return 0;
}
```

D:\Cristina\codeblocks\problema cluj\main.cpp

The result of the C++ program:



This is what happens in the C++ program: