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Fractions having a nice profile

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Introduction

A positive fraction is said to have a nice profile if it can be written as a sum of other fractions, all different, each one of the form 1/*p* with *p* a positive integer. For instance, the fraction 5/6 = 1/2 + 1/3 has a nice profile. There is a lot of questions that can be asked about this topic. For example, which fractions have a nice profile? Is it possible to find general and automatic methods to write any given fraction in this way? If a fraction has a nice profile, how many ways are there to break it down? What if we are interested in sums of two fractions only?

Our results

Theorem 1. *Any fraction of the type* "$\frac{1}{n}$"," $\frac{n+1}{n}$", $n \in \mathbb{N}^*$, " $\frac{2}{n}$", $n \in \mathbb{N}^*$ and $(2,n) = 1$,[1] *or* "$\frac{n+2}{n}$", $n \in \mathbb{N}^*_{\setminus\{1\}}$ and $(2,n) = 1$, *has a good profile, and it can be decomposed in infinitely many ways as a sum of "m" different fractions, having the numerator "1" , for* $m \geq 3$. *In particular,*

$$\textbf{I. } \boldsymbol{a}) \quad \mathbf{1} = \frac{\mathbf{1}}{\mathbf{1}\cdot\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}\cdot\mathbf{3}} + \cdots \ldots + \frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\cdot\boldsymbol{a}} + \frac{\mathbf{1}}{\boldsymbol{a}}, \ \textbf{ for } \boldsymbol{a} \textbf{ odd and } \boldsymbol{a} \geq \mathbf{3}$$

$$\textbf{I. b}) \quad \frac{\mathbf{1}}{\boldsymbol{n}} = \frac{\mathbf{1}}{\mathbf{1}\cdot\mathbf{2}\cdot\boldsymbol{n}} + \frac{\mathbf{1}}{\mathbf{2}\cdot\mathbf{3}\cdot\boldsymbol{n}} + \cdots \ldots + \frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\cdot\boldsymbol{a}\cdot\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{a}\cdot\boldsymbol{n}}, \textbf{ for } \boldsymbol{a} \textbf{ odd and } \boldsymbol{a} \geq \mathbf{3}$$

$$\textbf{II. a}) \quad \frac{\mathbf{2}}{\boldsymbol{n}} = \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}} = \frac{\mathbf{1}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\mathbf{1}\cdot\mathbf{2}\cdot\boldsymbol{n}} + \frac{\mathbf{1}}{\mathbf{2}\cdot\mathbf{3}\cdot\boldsymbol{n}} + \cdots \ldots \frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\cdot\boldsymbol{a}\cdot\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{a}\cdot\boldsymbol{n}},$$

for $\boldsymbol{a}$ odd, $\boldsymbol{a} \geq \mathbf{3}$ and $(\mathbf{2}, \boldsymbol{n}) = \mathbf{1}$

[1] (*m*,*n*) denotes the Greatest Common Divisor (GCD) of *m* and *n.*

$$\mathbf{II.b)}\quad \frac{n+1}{n}=1+\frac{1}{n}=\frac{1}{1}+\frac{1}{1\cdot 2\cdot n}+\frac{1}{2\cdot 3\cdot n}+\cdots\dots\frac{1}{(a-1)\cdot a\cdot n}+\frac{1}{a\cdot n},$$

for a odd and $a\geq 3, n\in\mathbb{N}^*$

$$\mathbf{II.c)}\quad \frac{n+2}{n}=1+\frac{2}{n}=\frac{1}{1}+\frac{1}{n}+\frac{1}{1\cdot 2\cdot n}+\frac{1}{2\cdot 3\cdot n}+\cdots\dots\frac{1}{(a-1)\cdot a\cdot n}+\frac{1}{a\cdot n},$$

for a odd, $a\geq 3, n\in\mathbb{N}^*_{\setminus\{1\}}$ and $(2,n)=1$

$$\mathbf{II.d)}\quad \frac{n-1}{n}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\dots+\frac{1}{(n-1)\cdot n},\text{for } n\geq 3$$

$$\mathbf{II.e)}\quad \frac{2n-1}{n}=\frac{1}{1}+\frac{n-1}{n}=\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\dots+\frac{1}{(n-1)\cdot n},\text{for n}\geq 3$$

Theorem 2. *Any fraction of the form $\frac{p}{q}$, where $q\geq 2,\ (p,q)=1$ and $p|q+1$* [2] *can be written as a sum of two different fractions having the numerator 1:*

$$\frac{p}{q}=\frac{1}{\frac{1+q}{p}}+\frac{1}{\frac{q(q+1)}{p}}$$

Proof of Theorem 1

I. FRACTIONS OF THE FORM "$\frac{1}{n}$", $n\in\mathbb{N}^*$

First, we tried to write number "1" as a sum of fractions with the numerator "1". We used that:

$$\frac{1}{n(n+1)}=\frac{(n+1)-n}{n(n+1)}=\frac{(n+1)}{n(n+1)}-\frac{n}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

For example:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3}=1\quad (\text{because}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}=1)$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4}=1\quad (\text{because}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}=1)$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5}=1$$

$$\frac{1}{1\cdot 2}+\mathbf{\frac{1}{2\cdot 3}}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\mathbf{\frac{1}{6}}=1$$

We obtained two equal fractions. In order that all the fractions are different, the last fraction must have the denominator odd, because the denominators of the rest of the fractions are even, different.

So

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\dots+\frac{1}{(a-1)\cdot a}+\frac{1}{a}=1,\text{for } a \text{ odd and } a\geq 3\qquad (1)$$

[2] $p|q$ is read as "p is a divisor of p".

Proof:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\ldots+\frac{1}{(a-1)\cdot a}+\frac{1}{a}=$$

$$\frac{2-1}{1\cdot 2}+\frac{3-2}{2\cdot 3}+\cdots\ldots+\frac{a-(a-1)}{(a-1)\cdot a}+\frac{1}{a}=$$

$$\frac{2}{1\cdot 2}-\frac{1}{1\cdot 2}+\frac{3}{2\cdot 3}-\frac{2}{2\cdot 3}+\cdots\ldots+\frac{a-1}{(a-1)\cdot a}-\frac{(a-1)}{(a-1)\cdot a}+\frac{1}{a}=$$

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots\ldots+\frac{1}{a-1}-\frac{1}{a}+\frac{1}{a}=1$$

For example: if $a=7$ we obtain:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\frac{1}{7}=1$$

: if $a=99$ we obtain:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{98\cdot 99}+\frac{1}{99}=1$$

Conclusion: The fraction "$\frac{1}{1}$" has a good profile and it can be decomposed in an infinity of ways.

If we multiply the relation (1) by "$\frac{1}{n}$", $n\in\mathbb{N}^*_{\setminus\{1\}}$, we obtain ;

$$\frac{\mathbf{1}}{\mathbf{1}\cdot\mathbf{2}\cdot\boldsymbol{n}}+\frac{\mathbf{1}}{\mathbf{2}\cdot\mathbf{3}\cdot\boldsymbol{n}}+\cdots\ldots+\frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\cdot\boldsymbol{a}\cdot\boldsymbol{n}}+\frac{\mathbf{1}}{\boldsymbol{a}\cdot\boldsymbol{n}}=\frac{\mathbf{1}}{\boldsymbol{n}},$$

for $\boldsymbol{a}$ odd and $\boldsymbol{a}\ \geq \mathbf{3}$

For example, if $n=7$, we obtain:

$$\frac{1}{7}=\frac{1}{1\cdot 2\cdot 7}+\frac{1}{2\cdot 3\cdot 7}+\cdots\ldots+\frac{1}{(a-1)\cdot a\cdot 7}+\frac{1}{a\cdot 7}$$

If $a=3$, then

$$\frac{1}{7}=\frac{1}{1\cdot 2\cdot 7}+\frac{1}{2\cdot 3\cdot 7}+\frac{1}{3\cdot 7}$$

If $a=5$, then

$$\frac{1}{7}=\frac{1}{1\cdot 2\cdot 7}+\frac{1}{2\cdot 3\cdot 7}+\frac{1}{3\cdot 4\cdot 7}+\frac{1}{4\cdot 5\cdot 7}+\frac{1}{5\cdot 7}$$

II. Fraction of the forms [1]

$\frac{\mathbf{2}}{\boldsymbol{n}}$, $\frac{\boldsymbol{n}+\mathbf{1}}{\boldsymbol{n}}$, $n\in\mathbb{N}^*$, $\frac{\boldsymbol{n}+\mathbf{2}}{\boldsymbol{n}}$, $n\in\mathbb{N}^*_{\setminus\{1\}}$, $\frac{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{n}}$, $n\in\mathbb{N}^*_{\setminus\{1,2\}}$, $\frac{\mathbf{2}\boldsymbol{n}-\mathbf{1}}{\boldsymbol{n}}$, $n\in\mathbb{N}^*_{\setminus\{1\}}$

We can also obtain:

a) $$\frac{\mathbf{2}}{\boldsymbol{n}}=\frac{\mathbf{1}}{\boldsymbol{n}}+\frac{\mathbf{1}}{\boldsymbol{n}}=\frac{\mathbf{1}}{\boldsymbol{n}}+\frac{\mathbf{1}}{\mathbf{1}\cdot\mathbf{2}\cdot\boldsymbol{n}}+\frac{\mathbf{1}}{\mathbf{2}\cdot\mathbf{3}\cdot\boldsymbol{n}}+\cdots\ldots\frac{\mathbf{1}}{(\boldsymbol{a}-\mathbf{1})\cdot\boldsymbol{a}\cdot\boldsymbol{n}}+\frac{\mathbf{1}}{\boldsymbol{a}\cdot\boldsymbol{n}},$$

for $\boldsymbol{a}$ odd, $\boldsymbol{a} \geq \boldsymbol{3}$ and $(\boldsymbol{2}, \boldsymbol{n}) = \boldsymbol{1}$

For example, if $n = 7$ we obtain:

$$\frac{2}{7} = \frac{1}{7} + \frac{1}{7} = \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \cdots \ldots \frac{1}{(a-1) \cdot a \cdot 7} + \frac{1}{a \cdot 7}$$

If $a = 3$ then

$$\frac{2}{7} = \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 7}$$

If $a = 5$ then

$$\frac{2}{7} = \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7}$$

b) $$\boldsymbol{\frac{n+1}{n} = 1 + \frac{1}{n} = \frac{1}{1} + \frac{1}{1 \cdot 2 \cdot n} + \frac{1}{2 \cdot 3 \cdot n} + \cdots \ldots \frac{1}{(a-1) \cdot a \cdot n} + \frac{1}{a \cdot n}},$$

for $\boldsymbol{a}$ odd and $\boldsymbol{a} \geq \boldsymbol{3}, \boldsymbol{n} \in \mathbb{N}^*$

For example, if $n = 7$ we obtain:

$$\frac{8}{7} = \frac{1}{1} + \frac{1}{7} = \frac{1}{1} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \cdots \ldots \frac{1}{(a-1) \cdot a \cdot 7} + \frac{1}{a \cdot 7}$$

If $a = 3$ then

$$\frac{8}{7} = \frac{1}{1} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 7}$$

If $a = 5$ then

$$\frac{8}{7} = \frac{1}{1} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7}$$

c) $$\boldsymbol{\frac{n+2}{n} = 1 + \frac{2}{n} = \frac{1}{1} + \frac{1}{n} + \frac{1}{1 \cdot 2 \cdot n} + \frac{1}{2 \cdot 3 \cdot n} + \cdots \ldots \frac{1}{(a-1) \cdot a \cdot n} + \frac{1}{a \cdot n}},$$

for $\boldsymbol{a}$ odd, $\boldsymbol{a} \geq \boldsymbol{3}, \boldsymbol{n} \in \mathbb{N}^*_{\setminus\{\boldsymbol{1}\}}$ and $(\boldsymbol{2}, \boldsymbol{n}) = \boldsymbol{1}$

For example, if $n = 7$ we obtain:

$$\frac{9}{7} = \frac{1}{1} + \frac{2}{7} = \frac{1}{1} + \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \cdots \ldots \frac{1}{(a-1) \cdot a \cdot 7} + \frac{1}{a \cdot 7}$$

If $a = 3$ then

$$\frac{9}{7} = \frac{1}{1} + \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 7}$$

If $a = 5$ then

$$\frac{9}{7} = \frac{1}{1} + \frac{1}{7} + \frac{1}{1 \cdot 2 \cdot 7} + \frac{1}{2 \cdot 3 \cdot 7} + \frac{1}{3 \cdot 4 \cdot 7} + \frac{1}{4 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7}$$

Conclusion: Any fraction of the form “$\frac{1}{n}$", "$\frac{n+1}{n}$", $n \in \mathbb{N}^*$, "$\frac{2}{n}$", $n \in \mathbb{N}^*$ and $(2, n) = 1$, or “$\frac{n+2}{n}$", $n \in \mathbb{N}^*_{\setminus\{1\}}$ and $(2, n) = 1$, has a good profile, and it can be decomposed in infinitely many ways as a sum of *“m”* different fractions, having the numerator “1” , for $m \geq 3$.

d) $$\frac{n-1}{n} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{(n-1) \cdot n}, \textbf{for } n \geq 3$$

Examples:

Fraction (n-1)/n= =1/a+1/b…	*1/a*	*1/b*	*1/c*	*1/a*	*1/b*	*1/c*
1/2	1/3	1/6	-	1/3	1/7	1/42
1/2	1/3	1/8	1/24	1/3	1/9	1/18
1/2	1/3	1/10	1/15	1/4	1/6	1/12
1/2	1/4	1/5	1/20	-	-	-
2/3	1/2	1/6	-	1/2	1/7	1/42
2/3	1/2	1/8	1/24	1/2	1/9	1/18
2/3	1/2	1/10	1/15	1/3	1/4	1/12
3/4	1/2	1/4	-	1/2	1/6	1/12
3/4	1/2	1/5	1/20	1/3	1/4	1/6
4/5	1/2	1/5	1/10	1/2	1/4	1/20
5/6	1/2	1/3	-	1/2	1/4	1/12
6/7	1/2	1/3	1/42	-	-	-

We prove this result by mathematical induction:

Base clause: Let’s prove the property works for $n = 2$: $\frac{1}{1 \cdot 2} = \frac{2-1}{2} = \frac{1}{2}$

Induction clause: We assume that the property holds for n, let’s prove it for $n + 1$.

We have:

$$\sum_{k=2}^{n} \frac{1}{k(k-1)} = \frac{n-1}{n}$$

Let’s prove:

$$\sum_{k=2}^{n+1} \frac{1}{k(k-1)} = \frac{n}{n+1}$$

$$\sum_{k=2}^{n+1} \frac{1}{k(k-1)} = \sum_{k=2}^{n} \frac{1}{k(k-1)} + \frac{1}{n(n+1)}$$

$$= \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n+1)(n-1)+1}{n(n+1)}$$

$$= \frac{n^2 - 1 + 1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}$$

Conclusion: The property respects the base clause ($n = 2$) and the induction clause so it is true for every $n \in \mathbb{N}^*_{\setminus\{1\}}$

For example: if $n = 7$ we obtain:

$$\frac{6}{7} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{6 \cdot 7}$$

: if $n = 11$ we obtain:

$$\frac{10}{11} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{10 \cdot 11}$$

e) $$\mathbf{\frac{2n-1}{n} = \frac{1}{1} + \frac{n-1}{n} = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{(n-1) \cdot n}, \text{for } n \geq 3}$$

For example: if $n = 7$ we obtain:

$$\frac{13}{7} = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{6 \cdot 7}$$

: if $n = 11$ we obtain:

$$\frac{21}{11} = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \ldots + \frac{1}{10 \cdot 11}$$

Proof of Theorem 2

Fractions of the type $\frac{1}{a} + \frac{1}{b} = \frac{p}{q}$, where q is a prime number, $(p,q)=1$ [2]

1) $\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$, **where $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \ldots \ldots \cdot p_n^{a_n}$** [3]

and $p_1, p_2, \ldots \ldots, p_n$ are different prime numbers and $a_1, a_2, \ldots \ldots, a_n \in \mathbb{N}^*$

Examples:

Fraction *1/n=1/a+1/b*	***1/a***	***1/b***	***1/a***	***1/b***
1/2	1/3	1/6		
1/3	1/4	1/12		
1/4	1/6	1/12	1/5	1/20
1/5	1/6	1/30		
1/6	1/7	1/42	1/8	1/24
1/6	1/9	1/18	1/10	1/15
1/7	1/8	1/56		

$bn + an = ab \Leftrightarrow$

$b(a-n) - an = 0 \mid + n^2 \Leftrightarrow$

$b(a-n)-an+n^2=n^2 \Leftrightarrow$
$b(a-n)-n(a-n)=n^2 \Leftrightarrow$

$$(\boldsymbol{a}-\boldsymbol{n})(\boldsymbol{b}-\boldsymbol{n})=\boldsymbol{n}^2\text{ , where }\boldsymbol{n}^2=\boldsymbol{p}_1^{2a_1}\cdot\boldsymbol{p}_2^{2a_2}\cdot\dots\dots\dots\cdot\boldsymbol{p}_n^{2a_n} \qquad \textbf{(2)}$$

n^2 has $N=(2a_1+1)(2a_2+1)\cdot\ldots\cdot(2a_n+1)$, natural divisors. As N is odd, the equation has $(2a_1+1)(2a_2+1)\cdot\ldots\ldots\ldots\cdot(2a_n+1)-1$ solutions, put in order, that is, a finite number of solutions.

a) If $n=p$ where "p" is a prime number

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{p} \qquad [4]$$

We obtain from the relation (2)
$(a-p)(b-p)=p^2 \Rightarrow$
$(a-p,b-p)\in\{(1,p^2),(p,p),(p^2,1)\}\Rightarrow$
$(a,b)\in\{(1+p,p^2+p),(2p,2p),(p^2+p,1+p)\}$
As $a\neq b\Rightarrow(a,b)\in\{(1+p,p(p+1)),(p(p+1),1+p)\}$

So, any fraction of the type "$\frac{1}{p}$", where "p" is a prime number, can be written as:

$$\frac{1}{p}=\frac{1}{p+1}+\frac{1}{p(p+1)}$$

For example:

$$\frac{1}{7}=\frac{1}{8}+\frac{1}{7\cdot 8}; \qquad \frac{1}{11}=\frac{1}{12}+\frac{1}{11\cdot 12}; \qquad \frac{1}{99}=\frac{1}{100}+\frac{1}{99\cdot 100}$$

b) If $n=p\cdot q$ where "p"," q" are different prime numbers:

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{pq}$$

From the relation (2), we obtain
$(a-pq)(b-pq)=p^2q^2\Rightarrow$

$$(a-pq,b-pq)\in\left\{\begin{matrix}(1,p^2q^2),(p,pq^2),(q,qp^2),(p^2,q^2),(q^2,p^2),(qp^2,q),\\(pq^2,p),(p^2q^2,1),(pq,pq)\end{matrix}\right\}$$

As $a\neq b\Rightarrow$

$$(a,b)\in\left\{\begin{matrix}(1+pq,p^2q^2+pq),(p+pq,pq^2+pq),(q+pq,qp^2+pq),(p^2+pq,q^2+pq),\\(q^2+pq,p^2+pq),(qp^2+pq,q+pq),(pq^2+pq,p+pq),(p^2q^2+pq,1+pq)\end{matrix}\right\}$$

We have obtained 8 ordered pairs of (a,b)
If $p,q\in\mathbb{N}^*$, the solutions we can obtain are several.
For example:

$$\frac{1}{2\cdot 3}=\frac{1}{7}+\frac{1}{6\cdot 7}; \qquad \frac{1}{2\cdot 3}=\frac{1}{2\cdot 4}+\frac{1}{2\cdot 3\cdot 4};$$
$$\frac{1}{2\cdot 3}=\frac{1}{3\cdot 3}+\frac{1}{2\cdot 3\cdot 3}; \qquad \frac{1}{2\cdot 3}=\frac{1}{2\cdot 5}+\frac{1}{3\cdot 5}$$

c) If $n = p^2$ where "p" is a prime number:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{p^2}$$

From the relation (2), we obtain

$(a - p^2)(b - p^2) = p^4 \Rightarrow$

$(a - p^2, b - p^2) \in \{(1, p^4), (p, p^3), (p^3, p), (p^2, p^2), (p^4, 1)\} \Rightarrow$

$(a, b) \in \left\{ \begin{matrix} (1 + p^2, p^4 + p^2), (p + p^2, p^3 + p^2), (p^3 + p^2, p + p^2), \\ (p^2 + p^2, p^2 + p^2), (p^4 + p^2, 1 + p^2) \end{matrix} \right\}$

As $a \neq b$

$$\Rightarrow (a, b) \in \left\{ \begin{matrix} (1 + p^2, p^2(p^2 + 1)), (p(p + 1), p^2(p + 1)), (p^2(p + 1), p(p + 1)), \\ (p^2(p^2 + 1), 1 + p^2) \end{matrix} \right\}$$

We obtained 4 ordered pairs of (a, b)

a) $\frac{1}{1+p^2} + \frac{1}{p^2(p^2+1)} = \frac{1}{p^2}$, for $p \geq 2$

b) $\frac{1}{p(p+1)} + \frac{1}{p^2(1+p)} = \frac{1}{p^2}$, for $p \geq 2$

For example:

$$\frac{1}{3^2} = \frac{1}{10} + \frac{1}{9 \cdot 10}; \qquad \frac{1}{3^2} = \frac{1}{3 \cdot 4} + \frac{1}{9 \cdot 4}$$

Conclusion: We can affirm that any fraction of the type: "$\frac{1}{n}$", $n \in \mathbb{N}^*_{\setminus\{1\}}$ can be written like this:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

Proof:

$$\frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{n}{n(n+1)} + \frac{1}{n(n+1)} = \frac{n+1}{n(n+1)} = \frac{1}{n}$$

2) $\frac{1}{a} + \frac{1}{b} = \frac{p}{q}$, where q is a prime number, $(p, q) = 1$;

$bq + aq = apb$

$a(bp - q) - bq = 0 / \cdot p$

$ap(bp - q) - pbq = 0 / + q^2$

$ap(bp - q) - pbq + q^2 = q^2$

$ap(bp - q) - q(bp - q) = q^2$

$(bp - q)(ap - q) = q^2$

$(ap - q, bp - q) \in \{(1, q^2), (q, q), (q^2, 1)\} \Rightarrow$

$(ap, bp) \in \{(1+q, q^2+q), (2q, 2q), (q^2+q, 1+q)\}$

As $a \neq b$, we obtain two ordered pairs of (a, b), only if "$p|q+1$"

$$(a,b) \in \{\left(\frac{q+1}{p}, \frac{q(q+1)}{p}\right), \left(\frac{q(q+1)}{p}, \frac{q+1}{p}\right)\}$$

How $q \neq 1 \Rightarrow \frac{1+q}{p} \neq \frac{q(q+1)}{p}$

If q is not a prime number, than q^2 has more than 3 divisors, so the equation can also have other solutions.

Conclusion: Any fraction of the type "$\frac{p}{q}$", where $q \geq 2$ and $(p, q) = 1$, can be written as a sum of two different fractions having the numerator "1", if "$p|q+1$".

$$\frac{p}{q} = \frac{1}{\frac{1+q}{p}} + \frac{1}{\frac{q(q+1)}{p}}$$

Particular cases:

a) **If $p = 2 \Rightarrow q = 2n+1, n \in \mathbb{N}^* \Rightarrow$**

$$\frac{2}{2n+1} = \frac{1}{n+1} + \frac{1}{(n+1)(2n+1)}$$

b) **If $p = 3 \Rightarrow q = 3n+2, n \in \mathbb{N}^* \Rightarrow$**

$$\frac{3}{3n+2} = \frac{1}{n+1} + \frac{1}{(n+1)(3n+2)}$$

c) **If $p = q+1 \Rightarrow$**

$$\frac{q+1}{q} = \frac{1}{1} + \frac{1}{q} \text{ (see II.)}$$

For example:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{2 \cdot 3}; \quad \frac{2}{5} = \frac{1}{3} + \frac{1}{3 \cdot 5}; \quad \frac{3}{5} = \frac{1}{2} + \frac{1}{2 \cdot 5}; \quad \frac{4}{15} = \frac{1}{4} + \frac{1}{4 \cdot 15}; \quad \frac{5}{14} = \frac{1}{3} + \frac{1}{3 \cdot 14};$$

Observation: There are fractions which can not be written as the sum of two different fractions having the numerator "1", because they doesn't fit the condition"$p|q+1$": [5]
:

$$\frac{1}{1}; \frac{4}{5}; \frac{3}{7}; \frac{5}{7}; \frac{6}{7}; \frac{5}{11}; \frac{8}{11}; \frac{9}{11}; \frac{10}{11}; \dots\dots$$

EDITION NOTES

[1] What do the five cases II.a – II.e have in common?

[2] If q is a prime number, then $(p,q)=1$ except if p is a multiple of q. But this is impossible, since $1/a+1/b < 1$ except for $a=b=2$, thus $p=q$ that is not interesting. Moreover, in the following q is sometimes not prime (cases III.b-III.c), so the statements are unclear

[3] .This is stated as a condition, but it is not because any natural number can be written in the form $p_1^{a_1} \dots p_n^{a_n}$.

[4] Why is this result different from those in Theorem 1?.

[5] Here you stress something different from what is proved above. You stress that if $p \mid q+1$ is not satisfied, then the fraction does not have a nice profile. But, in Theorem 1, only the inverse implication is proved. There are actually counterexamples. For, instance, the fraction 5/6 does fit that condition, but it can be written as 1/2 + 1/3.