Flower Beds and a Little Geometry

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The main decorative elements of a public park are: flower beds, water basins, artesian fountains, shops, trees, etc.

In the geometric structure of the parks the triangles, quadrilaterals, polygons and circles most often occur; obviously, the properties of these figures play an important role.

We intend to highlight the presence of flat geometry elements in the decoration of flower beds.(0)

1. Pappus' theorem and the decoration of parks

Let *MNP* be a scalene triangle. With a simple procedure, we will build two new triangles (fig. 1):

- the triangle $M'N'P'$, where $M' = \mathrm{sim}_M P$, $N' = \mathrm{sim}_N M$ and $P' = \mathrm{sim}_P N$ (the direct transform of the *MNP* triangle);

- the triangle $M''N''P''$, where $M'' = \mathrm{sim}_M N$, $N'' = \mathrm{sim}_N P$ and $P'' = \mathrm{sim}_P M$ (the inverse transform of the *MNP* triangle).

The direct or inverse order in which the vertices of the *MNP* triangle are traversed leads to the difference between the triangles $M'N'P'$ and $M''N''P''$.(1)

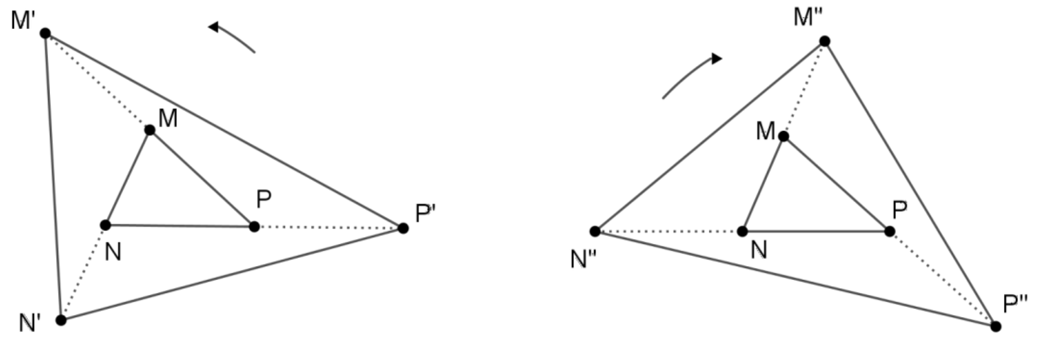


Fig. 1

1.1. Observation. The centroids of the triangles $M'N'P'$ and $M''N''P''$ coincide with the centroid of the triangle *MNP*.

The statement follows immediately using Pappus' well-known theorem(2) as we have:

$$\frac{\overline{NP'}}{\overline{PP'}}=\frac{\overline{PM'}}{\overline{MM'}}=\frac{\overline{MN'}}{\overline{NN'}}=2 \text{ and } \frac{\overline{NN''}}{\overline{PN''}}=\frac{\overline{PP''}}{\overline{MP''}}=\frac{\overline{MM''}}{\overline{NM''}}=\frac{1}{2}.$$

Being given the triangle *ABC*, which we denote T_0, we build the sequence of triangles:

$$(1) \quad T_0, T_1, \ldots, T_n, \ldots$$

as follows: T_1 is the direct transform of T_0, T_2 is the inverse transform of T_1, and, generally, T_{2k+1} is the direct transform of T_{2k}, and T_{2k+2} is the inverse transform of T_{2k+1}. The vertices of the triangle T_n are A_n, B_n, C_n (fig. 2).

1.2. Proposition. (i) *The triangles T_n, $n \in \mathbb{N}$, have the same centroid, point G, the centroid of the given triangle ABC.*

(ii) $\sigma(T_n)=7\sigma(T_{n-1})$, $n \geq 1$ *(where $\sigma(MNP)$ is the area of triangle MNP).*

(iii) *The triangles from the subsequence $T_0, T_2, \ldots, T_{2n}, \ldots$ have parallel sides and a similarity ratio of 7; this property is true for the subsequence $T_1, T_3, \ldots, T_{2n+1}, \ldots$, as well, of the sequence* (1).

Proof.

(i) Results from the observation 1.1.

(ii) Let's show that $\sigma(M'N'P')=7\sigma(MNP)$; Similarly, we show that $\sigma(M''N''P'')=7\sigma(MNP)$ (fig. 1).

Firstly, we have that $\sigma(M'PP') = 2\sigma(MNP)$, because $PP' = NP$ and the altitude from M' relative to PP' is double the altitude from *M* of the triangle *MNP*. We have, as well, $\sigma(P'NN') = \sigma(N'MM') = \sigma(MNP)$. As a result, $\sigma(M'N'P') = \sigma(M'PP') + \sigma(P'NN') + \sigma(N'MM') + \sigma(MNP) = 7\sigma(MNP)$. Finally, the affirmation (ii) is true.

(iii) Obviously(3), it is enough to show that the triangles *ABC* and $A_2B_2C_2$ have parallel sides and a similarity ratio of 7.

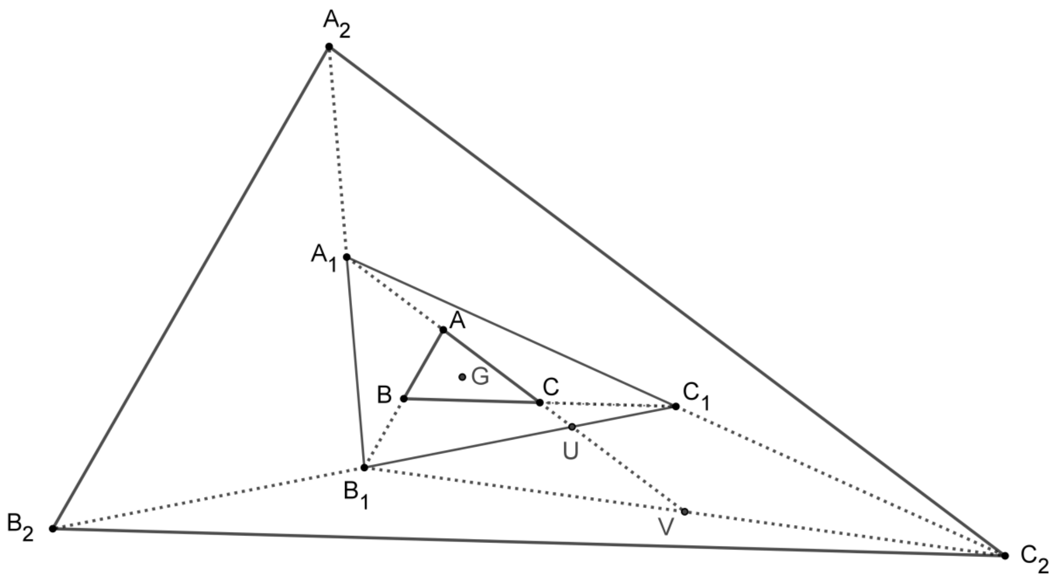


Fig. 2

Solution 1 (synthetic). We denote $U = AC \cap B_1C_1$ and $V = AC \cap B_1C_2$. According to Menelaus' theorem, applied to the triangle B_1C_1B and to the transversal $U - C - A$, we have:

$$\frac{UB_1}{UC_1} \cdot \frac{CC_1}{CB} \cdot \frac{AB}{AB_1} = 1 \Leftrightarrow \frac{UB_1}{UC_1} \cdot 1 \cdot \frac{1}{2} = 1 \Leftrightarrow UB_1 = 2UC_1 .$$

Then, with the same theorem applied to the triangle $C_2B_1C_1$ and to the transversal $V - U - A_1$, we obtain:

$$\frac{VB_1}{VC_2} \cdot \frac{A_1C_2}{A_1C_1} \cdot \frac{UC_1}{UB_1} = 1 \Leftrightarrow \frac{VB_1}{VC_2} \cdot 2 \cdot \frac{1}{2} = 1 \Leftrightarrow VB_1 = VC_2 .$$

So, A_1V is the middle line in the triangle $B_1A_2C_2$; thus, $A_1V \parallel A_2C_2$ and $AC \parallel A_2C_2$.

Analogously, we obtain that $AB \parallel A_2B_2$ and $BC \parallel B_2C_2$. The similarity ratio of the triangles $A_2B_2C_2$ and *ABC* is 7, because, according to the point (ii) of the proposition, we have $\sigma\left(A_2B_2C_2\right) = 7^2\sigma\left(ABC\right)$.

Solution 2 (complex numbers). We consider *xOy* a cartesian axis system in the plane and we denote the affix of a point *P* in the plane with $p \in \mathbb{C}$.

Because *A* is the middle of A_1C, we obtain that $2a = a_1 + c$. Thus, $a_1 = 2a - c$. Analogously, we get that $b_1 = 2b - a$ and $c_1 = 2c - b$.

Then, because A_1 is the middle of the segment B_1A_2, we obtain that $a_2 = 2a_1 - b_1$, therefore $a_2 = 2(2a-c)-(2b-a) = 5a - 2b - 2c$. Similarly, we get that $b_2 = 5b - 2c - 2a$ and $c_2 = 5c - 2a - 2b$. As a result, we have:

$$\frac{a_2 - b_2}{a - b} = \frac{b_2 - c_2}{b - c} = \frac{c_2 - a_2}{c - a} = 7 \in \mathbb{R} ,$$

which proves that $AB \parallel A_2B_2$, $BC \parallel B_2C_2$, $CA \parallel C_2A_2$ and that $\frac{A_2B_2}{AB} = \frac{B_2C_2}{BC} = \frac{C_2A_2}{CA} = 7$, meaning that the triangles $A_2B_2C_2$ and *ABC* have parallel sides and a similarity ratio of 7.

We ask ourselves whether the direct and inverse transformations defined above are inversible. The answer is yes, as the following result shows.

1.3. Proposition. *Being given a triangle* $A^*B^*C^*$ *the following affirmations are true:*

(i) *There exists a unique triangle* $A'B'C'$ *which transforms into* $A^*B^*C^*$ *through a direct transformation.*

(ii) *There exists a unique triangle* $A''B''C''$ *which transforms into* $A^*B^*C^*$ *through an inverse transformation.*

Proof.

(i) *Solution 1 (synthetic).* Let $A'B'C'$ be a triangle which is transformed into $A^*B^*C^*$ through a direct transformation. We adopt the notations: $D = A^*A' \cap B^*C^*$, $E = B^*B' \cap C^*A^*$, $F = C^*C' \cap A^*B^*$ (fig. 3). We'll show that $DC^* = \frac{1}{3}B^*C^*$, $EA^* = \frac{1}{3}C^*A^*$ and $FB^* = \frac{1}{3}A^*B^*$.

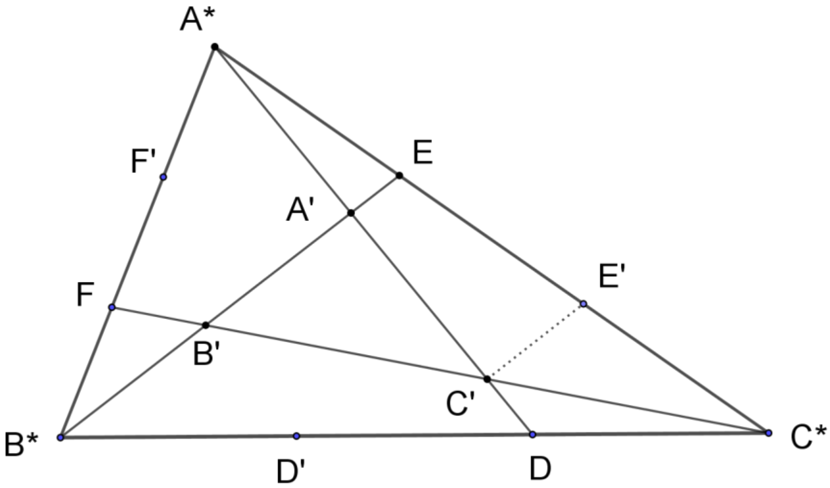


Fig. 3

Indeed, applying the Menelaus theorem(4) to the triangle $C^*C'A^*$ and to the transversal $B'-A'-E$, we have:

$$\frac{B'C^*}{B'C'}\cdot\frac{A'A^*}{A'C'}\cdot\frac{EA^*}{EC^*}=1\Leftrightarrow 2\times 1\times\frac{EA^*}{EC^*}=1\Leftrightarrow EC^*=2EA^*\Leftrightarrow EA^*=\frac{1}{3}C^*A^*;$$

in the same way, the other two equalities are obtained. So, if $A'B'C'$ exists, the position of the points *D*, *E*, *F* on the sides of $A^*B^*C^*$ is given by the relations above. This implies the uniqueness of the triangle.

In order to show the existence of a triangle $A'B'C'$ with the required property, we divide the sides B^*C^*, C^*A^* and A^*B^* into three equal parts through the points *D* and D', *E* and E', respectively *F* and F'. The lines A^*D, B^*E and C^*F determine the triangle $A'B'C'$ (fig. 3). We will prove that its direct transform is the given triangle $A^*B^*C^*$ and the proof will be completed. We use the theorem of Menelaus again: the triangle A^*DC^* and the transversal $B^*-A'-E$ lead to

$$\frac{B^*D}{B^*C^*}\cdot\frac{EC^*}{EA^*}\cdot\frac{A'A^*}{A'D}=1\Leftrightarrow\frac{2}{3}\cdot 2\cdot\frac{A'A^*}{A'D}=1\Leftrightarrow\frac{A'D}{A'A^*}=\frac{4}{3}\Leftrightarrow A^*D=\frac{7}{3}A'A^*,$$

while the triangle A^*B^*D and the transversal $C^*-C'-F$ lead to

$$\frac{C^*B^*}{C^*D}\cdot\frac{C'D}{C'A}\cdot\frac{FA^*}{FB^*}=1\Leftrightarrow\frac{C'D}{C'A}=\frac{1}{6}\Leftrightarrow A^*D=\frac{7}{6}C'A^*.$$

The relations $A^*D=\frac{7}{3}A'A^*$ and $A^*D=\frac{7}{6}C'A^*$, involve $C'A^*=2A'A^*$, so $C'A'=A'A^*$. Similarly, we have $A'B'=B'B^*$, $B'C'=C'C^*$, which means that $A^*B^*C^*$ is the direct transform of the triangle $A'B'C'$.

Solution 2 (complex numbers). Let $p \in \mathbb{C}$ be the affix of the point *P* in a cartesian axis system *xOy*. If the direct transform of the triangle $A'B'C'$ is $A^*B^*C^*$, we have:

$$2a' = c' + a^*,\ 2b' = a' + b^*,\ 2c' = b' + c^*.$$

Solving this system in relation to a', b', c', we obtain:

$$a' = \frac{1}{7}\left(4a^* + 2c^* + b^*\right),\ b' = \frac{1}{7}\left(4b^* + 2a^* + c^*\right),\ c' = \frac{1}{7}\left(4c^* + 2b^* + a^*\right).$$

So, if $A'B'C'$ exists, then it is unique (with the affixes given by the previous formulas). Moreover, it can be immediately verified, by calculation, that the triangle $A'B'C'$ with the above affixes has the direct transform $A^*B^*C^*$, which completes the statement (i).

(ii) This statement is proved analogously(5). The triangle $A''B''C''$, whose direct transformation is $A^*B^*C^*$, has the vertices $A'' = A^*D' \cap C^*F'$, $B'' = B^*E' \cap A^*D'$ and $C'' = C^*F' \cap B^*E'$.

1.4. Observation. The synthetic solution offers the way of construction (with the ruler and the compass) of the triangles $A'B'C'$ and $A''B''C''$ when the triangle $A^*B^*C^*$ is given. That is, the sides of the triangle $A^*B^*C^*$ are divided into three equal parts by the points $D, D' \in B^*C^*$, $E, E' \in C^*A^*$ and $F, F' \in A^*B^*$. The lines A^*D, B^*E and C^*F are drawn in order to obtain the triangle $A'B'C'$ and, finally, the lines A^*D', B^*E' and C^*F' are drawn in order to obtain the triangle $A''B''C''$.

The previous theoretical considerations may be the source of various applications in the decoration of public gardens. Fig. 4 is an example in this case.

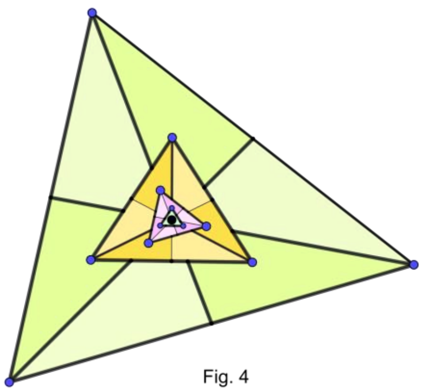


Fig. 4

Some explanations and observations regarding Fig. 4 are required:

1) Only four triangles were used : T_0, T_1, T_2, T_3; all are equilateral, because T_0 was equilateral. Starting with a scalene triangle, we get a much more varied line game. In observation 1.4. the practical way of building the figure is indicated.

2) At the vertices of the triangles, decorative trees or statues can be planted.

3) In the centre of the triangle T_0 a kiosk or an artesian fountain can be built.

4) Some of the segments in the figure can be alleys and other bushes limiting the flower beds. The segments, that are not on sides, are on the medians of the triangles and are directed to the centre of the figure (Proposition 1.2, (i)).

1.5. Observation. The proposition 1.3 and the observation 1.4 make it possible to adapt the decorative motive to the shape and size of the terrain. Starting with a triangle that takes into account the terrain, inside triangles can easily be built (calculations of the areas of the surfaces and the lengths of segments based on the proposition 1.2 can be made). Moreover, if the presence on the grounds of natural elements or other objects requires an initial triangle to be considered, new triangles can be constructed both inside and outside it using the transformations introduced and their inversions.

2. Conway's circle and flower beds

2.1. Proposition - Conway's circle. *The sides of a triangle ABC are extended considering the points* $A_1, B_2 \in AB - [AB]$, $B_1, C_2 \in BC - [BC]$, $C_1, A_2 \in AC - [AC]$ *such that* $AA_1 = AA_2 = a$, $BB_1 = BB_2 = b$ *and* $CC_1 = CC_2 = c$, *where we denoted* $AB = c$, $BC = a$, $CA = b$. *Then, the points* $A_1, A_2, B_1, B_2, C_1, C_2$ *belong to a circle with the centre in the centre of the circle inscribed in the triangle ABC. This circle is called the Conway circle of the triangle ABC (fig. 5).*

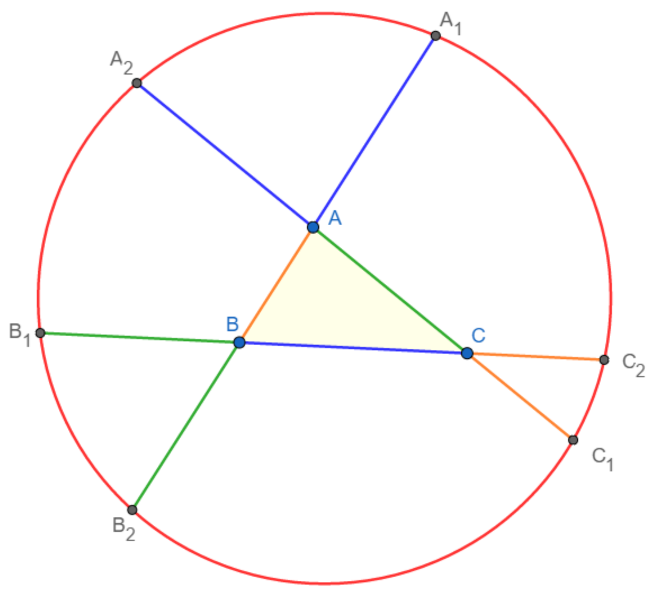


Fig. 5

2.2. Proposition - A generalization of Conway's circle. *If ABC is a triangle and the points* $A_1, B_2 \in AB-[AB]$, $B_1, C_2 \in BC-[BC]$, $C_1, A_2 \in AC-[AC]$ *such that* $AA_1 = AA_2 = k+a$, $BB_1 = BB_2 = k+b$ *and* $CC_1 = CC_2 = k+c$, *where we noted* $AB = c, BC = a, CA = b$ *and k is a real number so that* $k > \max(-a,-b,-c)$, *then* $A_1, A_2, B_1, B_2, C_1, C_2$ *appear on a circle with the centre at the centre of the circle inscribed in the triangle ABC (fig. 6).*

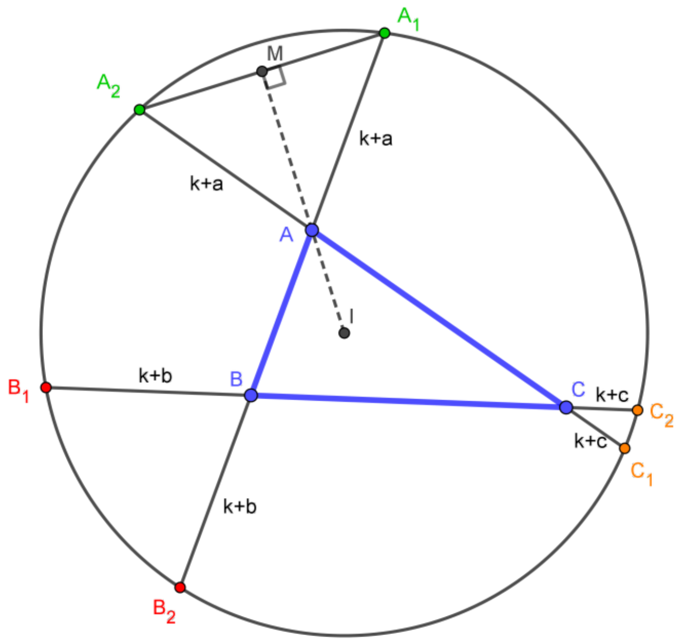


Fig. 6

Proof.

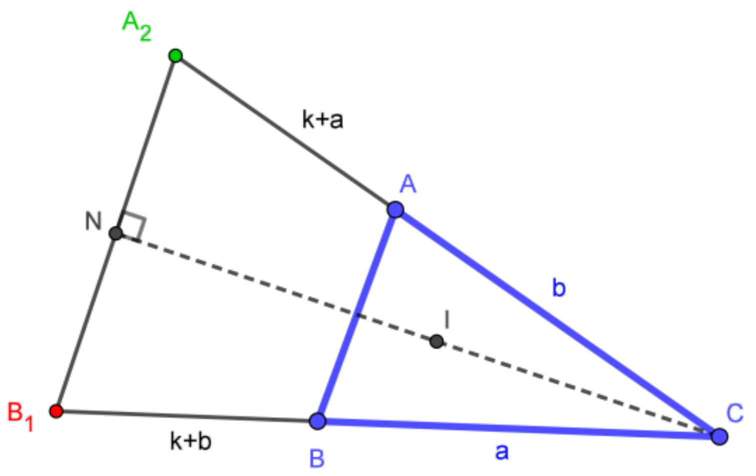
Because $k > \max(-a,-b,-c)$, we obtain that $k+a>0$, $k+b>0$ and $k+c>0$. Therefore, the construction made makes sense.

We denote *M* the middle of the segment A_1A_2 and *I* the centre of the circle inscribed in the triangle *ABC*.

In the triangle AA_1A_2, *AM* is a median and $AA_1 = AA_2 = k+a$, so, *AM* is the perpendicular bisector of the segment AA_1 and the bisector of the angle $\angle A_1AA_2$.

Because $\angle A_2AM = \frac{\angle A_1AA_2}{2} = \frac{\angle BAC}{2} = \angle IAC$, we get that $I \in AM$, the perpendicular bisector of A_1A_2, from which we obtain that $IA_1 = IA_2$. Similarly, $IB_1 = IB_2$ and $IC_1 = IC_2$.

We consider *N* the middle of the segment A_2B_1.



In the triangle A_2B_1C, we have $NA_2 = NB_1$ and $A_2C = A_2A + AC = k + a + b = B_1B + BC = B_1C$. Therefore, *CN* is the perpendicular bisector of A_2B_1 and the bisector of $\angle A_2CB_1$. Because the angle $\angle A_2CB_1$ is the angle $\angle ACB$ and *I* is the centre of the circle inscribed in the triangle *ABC*, we obtain that $I \in CN$. Thus, as *CN* is the perpendicular bisector of A_2B_1, we find that $IA_2 = IB_1$. Analogously, we get the relations $IB_2 = IC_1$ and $IC_2 = IA_1$ (fig. 7).

Fig. 7

So, we obtained that $IA_1 = IA_2 = IB_1 = IB_2 = IC_1 = IC_2$. Therefore, the points $A_1, A_2, B_1, B_2, C_1, C_2$ are on a circle with the centre in *I*, the centre of the circle inscribed in the triangle *ABC.*

For $k = 0 > \max(-a, -b, -c)$, we obtain Conway's Circle.

The theoretical considerations above may have various applications in the design and planning of public parks. Here are some examples.

In figure 8 we can see the plan of a circular public park. Lines are alleys, and the middle triangle is a central area with a circular fountain, which is tangent to the sides of the triangle. Thus, the centre of the fountain is the centre of the circle inscribed in the triangle.

In the vertices of the triangle, we can place statues, as well as in the centre of the fountain, and in the other points in the figure we lay trees on both sides of the alleys. Between the statues of the triangle’s vertices and the trees on the first circle is a distance equal to the side of the triangle, which we consider equilateral for symmetry. Between the trees on the alleys the distances are equal to any length is suitable for the land we have.

We notice that the trees on the first circle are there because they respect the terms of proposition 2.1. Therefore, they are on the Conway circle of the central triangle. As we have shown, it has the centre in the centre of the circle inscribed in the triangle, that is, the centre of the fountain. The next sets of trees are concyclic, respecting the terms of sentence 2.2. The circles on which they are located are also centred in the centre of the fountain. The park is therefore made up of various concentric circles.

In the green areas we can arrange green spaces and the coloured areas can be decorated with different flower beds.

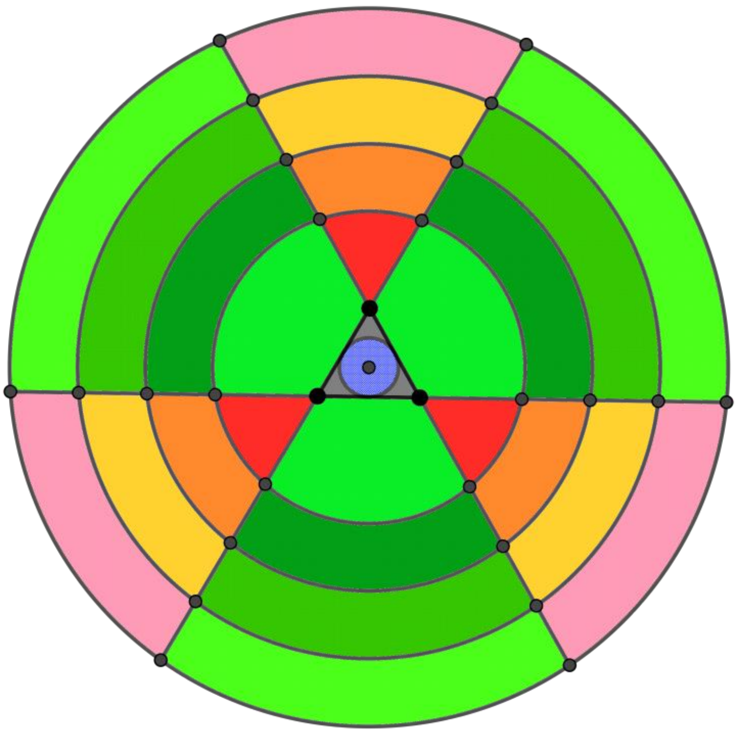


Fig. 8

If instead of an equilateral triangle, a right isosceles triangle is considered, the flower bed would resemble fig. 9. (propositions 2.1 and 2.2 are used in the same way again).

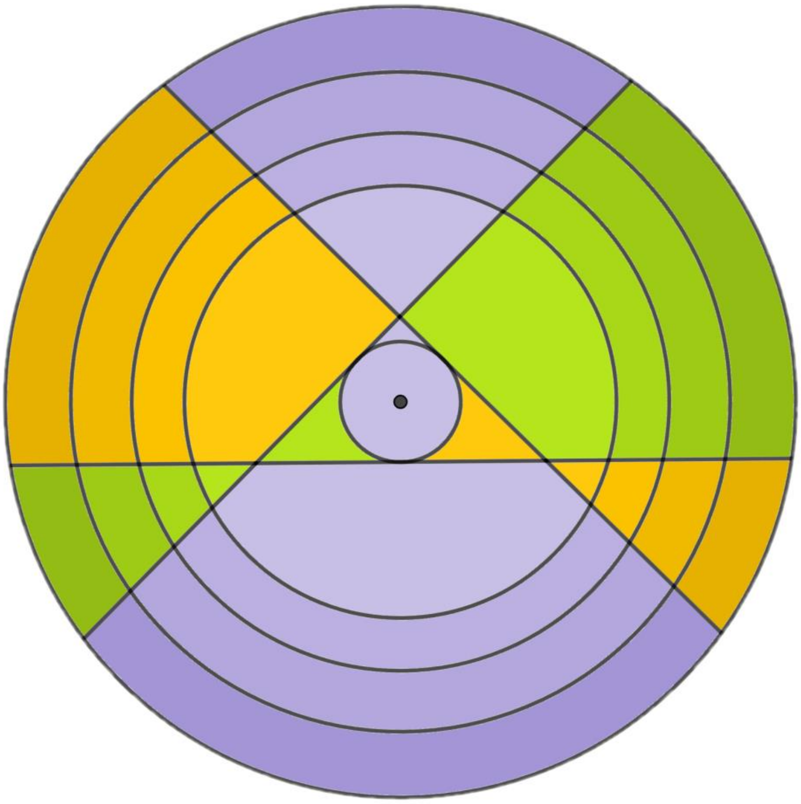


Fig. 9

Another application of the proposition 2.2 is the next flower bed, as imagined here in front of a palace:

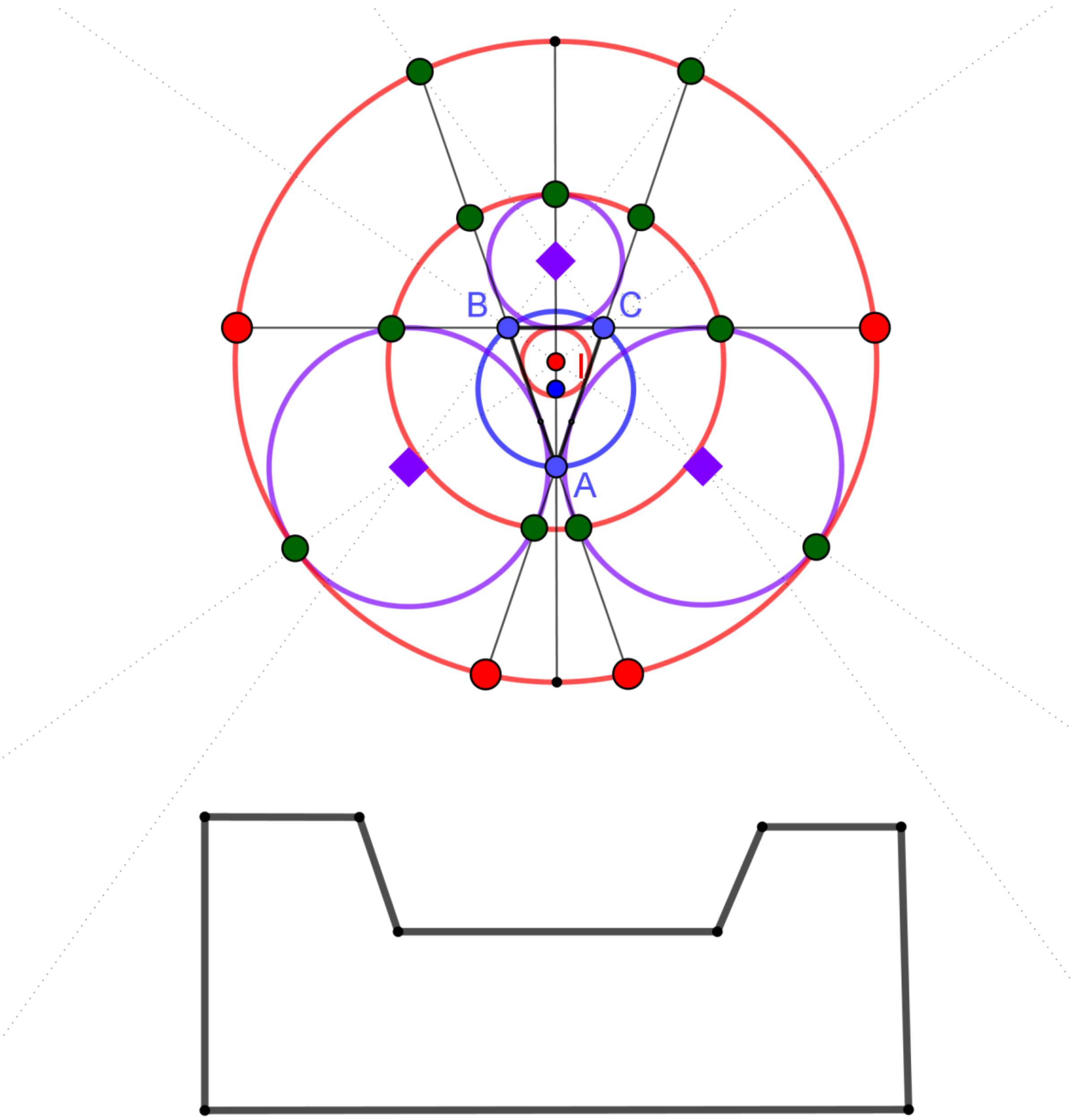


Fig. 10

The flower bed has an isosceles triangle as the starting point, and then continues with the inscribed and escribed circles(6) (red and purple), and two more red circles with the centre in *I* where *I* is the centre of the inscribed circle and the two red circles are tangent with the *A*-escribed circle or the *B*-escribed circle and the *C*-escribed circle. These Circles are *k*-Conway (proposition 2.2). Finding the value of *k* is necessary for each of the two circles if we want to construct them on the field. Next, we are going to look for the values of *k* for both circles.

Let $BC = a$, $AB = AC = b$. In order to simplify the calculations, we will consider the case: $a = 10x$, $b = 13x$, where *x* is a positive real number. Let I_a, I_b, I_c be the centres of the A-escribed, B-escribed and C-escribed circles, and r_a, r_b, r_c their radii. The radius of the inscribed circle will be r. We consider $D \in AC$, $E \in BC$, $F \in AC$ such that $ID \perp AC$, $IE \perp BC$, $I_cF \perp AC$. The intersection points of the big circles with the sides of the triangle are denoted according to fig. 11. Next, we look for k and k' such that the smaller red circle is k-Conway for ABC and tangent with the A-escribed circle, and the bigger one is k'-Conway for ABC and tangent with the B-escribed and C-escribed circles.

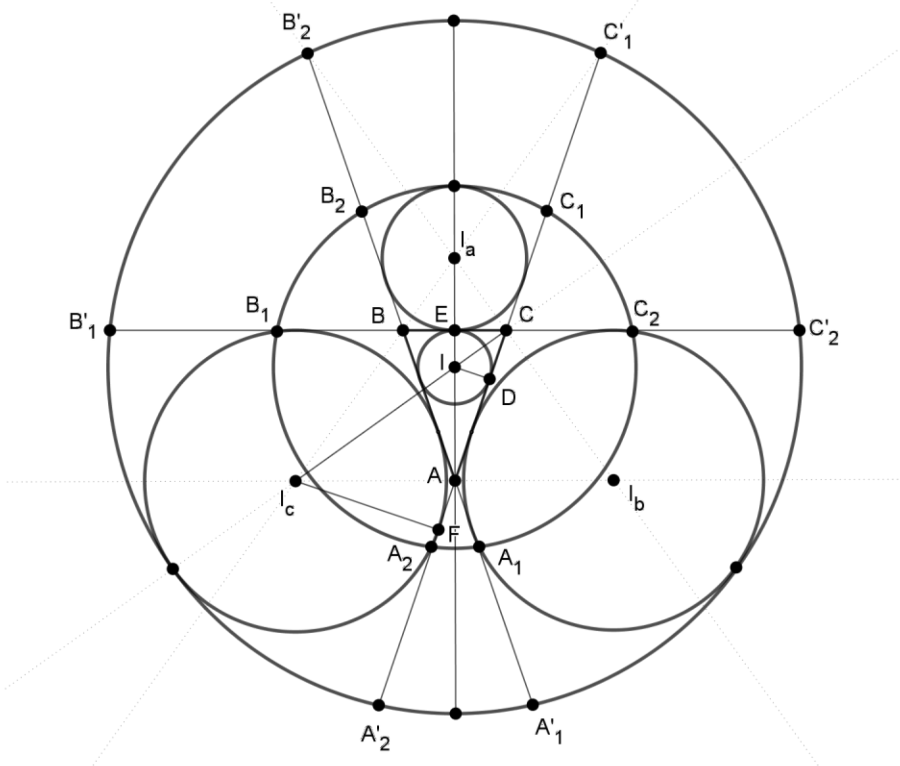


Fig. 11

Since ABC is isosceles, we have $CD = EC = \frac{a}{2} = 5x$. We know that(7) $r = \frac{A_{ABC}}{p} = \frac{10}{3}x$, $r_a = \frac{A_{ABC}}{p-a} = \frac{15}{2}x$, $r_b = r_c = \frac{A_{ABC}}{p-b} = 12x$, $II_a = r + r_a = \frac{65}{6}x$. Since C_1 is on the k-Conway circle, we get $DC_1 = DC + CC_1 = 5x + k + 13x = 18x + k$, which gives us $IC_1 = \sqrt{ID^2 + DC_1^{\,2}} = \sqrt{\left(\frac{10}{3}x\right)^2 + \left(18x + k\right)^2}$. The tangent condition between the k-Conway circle and the A-escribed circle is equivalent to $II_a = IC_1 - r_a$, which, with the proper replacements gives us $k = \left(15\sqrt{13} - 18\right)x$.

Similarly, $IC_1' = \sqrt{ID^2 + DC_1'^2} = \sqrt{\left(\frac{10}{3}x\right)^2 + (18x+k')^2}$, and $IC = \sqrt{ID^2 + DC^2} =$ $= \sqrt{\left(\frac{10}{3}x\right)^2 + (5x)^2} = \frac{5}{3}x\sqrt{13}$. Since *ID* and I_cF are perpendicular on *AC*, applying triangle similarity to *CID* and CI_cF, we get:

$$II_c = \frac{r_c - r}{r} CI = \frac{13}{3}x\sqrt{13}.$$

And such, the tangent condition between the k'-Conway and *C*-escribed circles (also with *B*-escribed) is equivalent with $II_c = IC_1' - r_c$, which, with the proper replacements gives us $k' = \left(4\sqrt{13} - 5\right)x$.

3. Quadrilaterals and park management

Let *ABCD* be a quadrilateral. Using a simple procedure, we will build two new quadrilaterals (fig. 12):

- $A'B'C'D'$, where $A' = \mathrm{sim}_A D$, $B' = \mathrm{sim}_B A$, $C' = \mathrm{sim}_C B$ and $D' = \mathrm{sim}_D C$ (<u>the direct transform</u> of *ABCD*);
- $A''B''C''D''$, where $A'' = \mathrm{sim}_A B$, $B'' = \mathrm{sim}_B C$, $C'' = \mathrm{sim}_C D$ and $D'' = \mathrm{sim}_D A$ (<u>the inverse transform</u> of *ABCD*).

The transformation order, direct or invers, in which the vertices of the *ABCD* quadrilateral are extended leads to the difference between the quadrilaterals $A'B'C'D'$ and $A''B''C''D''$.

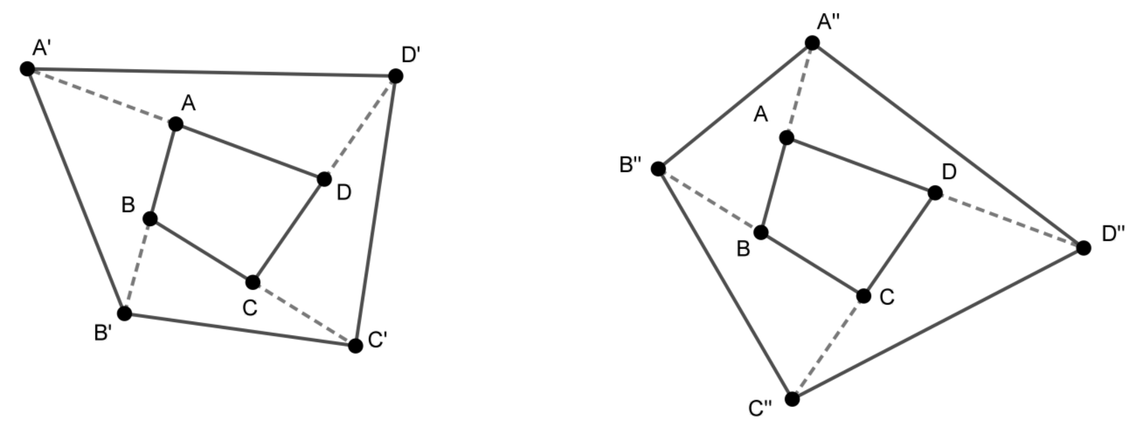


Fig. 12

3.1. Observation. The centre of mass of the new quadrilaterals $A'B'C'D'$ and $A''B''C''D''$ is also the centre of mass of the original quadrilateral *ABCD*.

Demonstration.

Considering the coordinate system *xOy* in the plane and denoting a,b,c,d,a',b',c',d' the affixes of A,B,C,D,A',B',C' respectively D'.

From the definition of the direct transform of a quadrilateral we get $2a = a' + d$, which gives us $a' = 2a - d$. Similarly, we find that $b' = 2b - a$, $c' = 2c - b$ and $d' = 2d - c$.

It is well known that the centre of mass for *ABCD* has the affix $\frac{a+b+c+d}{4}$, and that the centre of mass for $A'B'C'D'$ has the affix $\frac{a'+b'+c'+d'}{4}$. But,

$$\frac{a'+b'+c'+d'}{4} = \frac{2a-d+2b-a+2c-b+2d-c}{4} = \frac{a+b+c+d}{4}.$$

And so, quadrilaterals *ABCD* and $A'B'C'D'$ have the same centre of mass. Similarly, we find that quadrilaterals *ABCD* and $A''B''C''D''$ also have the same centre of mass.

(8)

3.2. Proposition. *If $ABCD$ is a convex orthodiagonal quadrilateral with equal diagonals and $A'B'C'D'$, $A''B''C''D''$ are the direct and inverse transforms, then quadrilaterals $A'B'C'D'$ and $A''B''C''D''$ are also orthodiagonal with equal diagonals (fig. 13).*

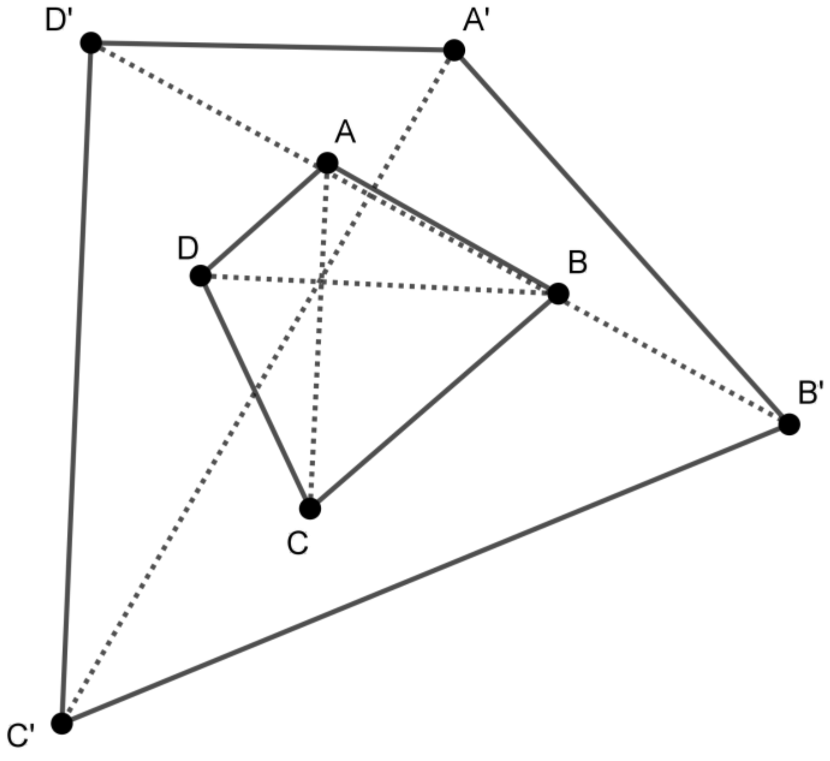


Fig. 13

Proof.

Considering the coordinate system *xOy* in the plane and denoting a,b,c,d,a',b',c',d' the affixes of A,B,C,D,A',B',C' respectively D'.

We can consider, without restricting the generality, that $a=0$ and $c=1$. Let $b=m+ni$, where $m,n\in\mathbb{R}$. We can also consider that point B is under the *AC* axis, and point *D* is positioned above(9).

That way, the line AC coincides with axis *Ox* and $AC=1$. Since $ABCD$ is orthodiagonal with equal diagonals, this means that $BD=1$ and *BD* is parallel with the axis *Oy*. This way, the real part d equals the real part of *b*. Since $BD=1$, this means $d=m+(n+1)\cdot i$, where $n\in(-1,0)$, since B is under AC and D is above.

We have $A'=sim_A D\Rightarrow a=\dfrac{d+a'}{2}\Rightarrow a'=2a-d=(-m)+(-n-1)i$. Similarly, we get $b'=2b-a=2m+2ni$, $c'=2c-b=(2-m)+(-n)i$ and $d'=2d-c=(2m-1)+(2n+2)i$.

And thus,

$$A'C'=|a'-c'|=\left|((-m)+(-n-1)i)-((2-m)+(-n)i)\right|=|-2-i|=\sqrt{(-2)^2+(-1)^2}=\sqrt{5};$$

$$B'D' = |b' - d'| = |(2m + 2ni) - ((2m-1) + (2n+2)i)| = |1 - 2i| = \sqrt{1^2 + (-2)^2} = \sqrt{5};$$

Therefore $A'C' = B'D'$.

$$A'C' \perp B'D' \Leftrightarrow \frac{a' - c'}{b' - d'} \in \square - \square \Leftrightarrow \frac{-2-i}{1-2i} \in \square - \square \Leftrightarrow \frac{(-2-i)(1+2i)}{(1-2i)(1+2i)} \in \square - \square \Leftrightarrow$$

$\Leftrightarrow \frac{-2-4i-i-2i^2}{5} \in \square - \square \Leftrightarrow -i \in \square - \square$, which is true(10).

In conclusion, $A'B'C'D'$ is orthodiagonal with equal diagonals. Similarly, we prove that $A''B''C''D''$ is orthodiagonal with equal diagonals.

A park built by directly transforming twice an orthodiagonal quadrilateral field with equal diagonals *ABCD* would look similar to the one in fig. 14, where both $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are orthodiagonal quadrilaterals with equal diagonals.

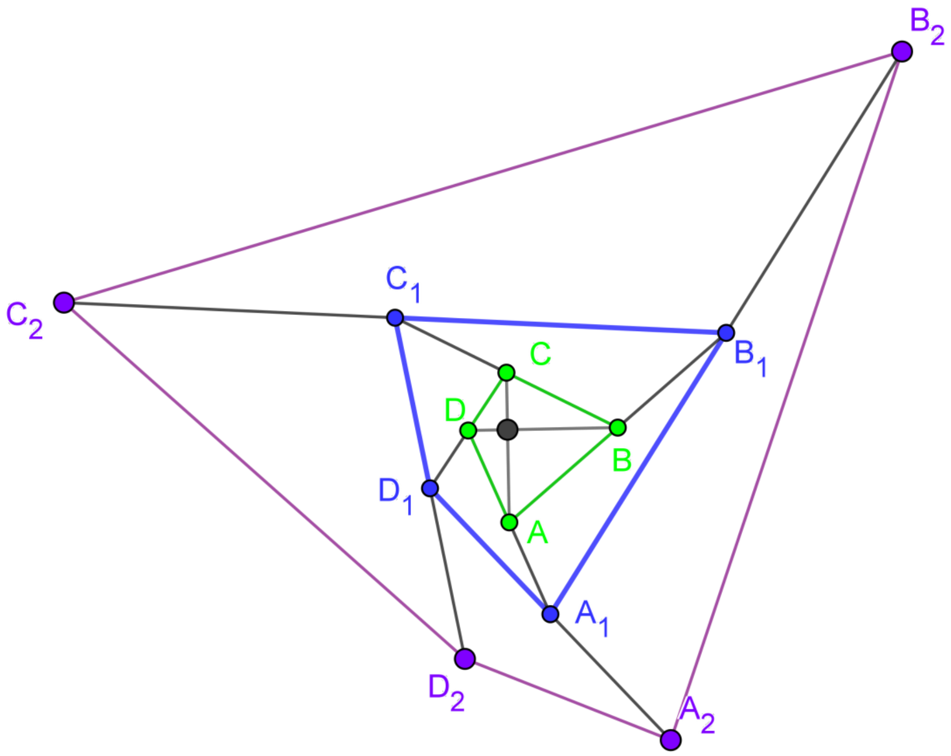


Fig. 14

3.3. Proposition. *Let ABCD be a convex quadrilateral and O the intersection point of the diagonals. Exterior to ABCD we build the isosceles triangles* AS_1B, BS_2C, CS_3D, DS_4A, *with bases AB, BC, CD, and respectively DA, such that*(11):

$$m(\angle AS_1B)+m(\angle AOB)=m(\angle BS_2C)+m(\angle BOC)=$$
$$=m(\angle CS_3D)+m(\angle BOD)=m(\angle DS_4A)+m(\angle DOA)=180^\circ.$$

We have the following properties:

a) The lines AC, BD, S_1S_3 and S_2S_4 intersect in the same point;

b) The lines S_1S_3 and S_2S_4 are perpendicular;

c) $S_1O=\dfrac{OA+OB}{\sqrt{2(1+\cos\alpha)}}$, $S_2O=\dfrac{OB+OC}{\sqrt{2(1+\cos\beta)}}$, $S_3O=\dfrac{OC+OD}{\sqrt{2(1+\cos\alpha)}}$ *and* $S_4O=\dfrac{OD+OA}{\sqrt{2(1+\cos\beta)}}$, *where* $\alpha=m(\angle AOB)$ *and* $\beta=m(\angle BOC)\left(\alpha+\beta=180^\circ\right)$.

d) If the quadrilateral ABCD is orthodiagonal, then $S_1S_3=S_2S_4$

e) The quadrilateral ABCD is a parallelogram if and only if the quadrilateral $S_1S_2S_3S_4$ *is a rhombus.*

f) The quadrilateral ABCD is a rhombus if and only if $S_1S_2S_3S_4$ *is a square.*(12)

Proof.

a) Because $m(\angle AS_1B)+m(\angle AOB)=180^\circ$, it is known(13) that the quadrilateral $AOBS_1$ is inscribable, so $m(\angle AOS_1)=m(\angle ABS_1)$ and $m(\angle BAS_1)=m(\angle BOS_1)$. Next, knowing that ABS_1 is an isosceles triangle with the base AB, we get $m(\angle AOS_1)=m(\angle BOS_1)$, which means that the ray OS_1 is the interior angle bisector of the AOB angle. Similarly, we show that OS_2 is the interior angle bisector for the angle BOC, OS_3 is the interior angle bisector for the angle COD and OS_4 is the interior angle bisector for the angle DOA. It is known that the interior angle bisectors of two opposite angles are opposite rays. So, the points S_1, O, S_3 are collinear and so are the points S_2, O, S_4 as well (fig. 15).

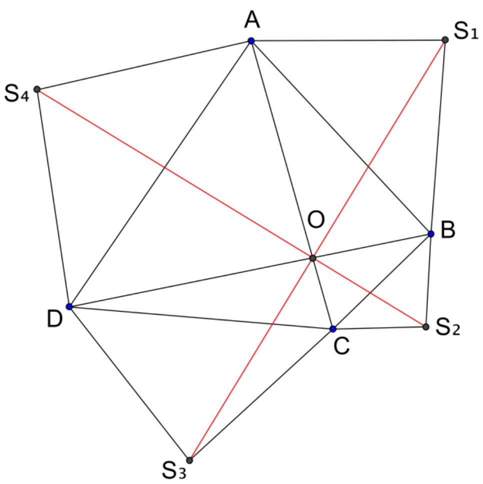


Fig. 15

b) Considering what we have already proved in the previous point and that the interior angle bisector and the exterior angle bisector are perpendicular, we now know that the lines S_1S_3 and S_2S_4 are perpendicular.

c) From Ptolemy's theorem applied to the inscribable quadrilateral $AOBS_1$, we find that:

$$S_1O \cdot AB = S_1B \cdot OA + S_1A \cdot OB$$

which, considering that $S_1A = S_1B$, can be written as:

$$(1) \qquad S_1O \cdot AB = S_1A \cdot (OA + OB).$$

By applying the law of cosines in the triangle AS_1B, we get:

$$AB^2 = S_1A^2 + S_1B^2 - 2S_1A \cdot S_1B \cdot \cos(\angle AS_1B) = S_1A^2 \cdot (2 + 2\cos\alpha).$$

or

$$(2) \qquad AB = S_1A\sqrt{2(1+\cos\alpha)}.$$

From the relations (1) and (2) we find that $S_1O = \dfrac{OA+OB}{\sqrt{2(1+\cos\alpha)}}$. The other three relations can be proved similarly.

d) If $ABCD$ is an orthodiagonal quadrilateral then $\cos\alpha = \cos\beta = 0$, and so, considering the relations from point c), we have:

$$S_1S_3 = S_1O + S_3O = \frac{OA+OB}{\sqrt{2}} + \frac{OC+OD}{\sqrt{2}} = \frac{AC+BD}{\sqrt{2}} = \frac{OB+OC}{\sqrt{2}} + \frac{OA+OD}{\sqrt{2}} = S_2O + S_4O = S_2S_4.$$

e) The quadrilateral $ABCD$ is a parallelogram if and only if

$$(3) \qquad AO = OC \text{ and } BO = OD.$$

The relations at (3) are equivalent with

$$OA + OB = OC + OD \text{ and } OB + OC = OD + OA,$$

or

$$\frac{OA+OB}{\sqrt{2(1+\cos\alpha)}} = \frac{OC+OD}{\sqrt{2(1+\cos\alpha)}} \text{ and } \frac{OB+OC}{\sqrt{2(1+\cos\beta)}} = \frac{OA+OD}{\sqrt{2(1+\cos\beta)}},$$

or

$$(4) \qquad S_1O = S_3O \text{ and } S_2O = S_4O.$$

Considering the perpendicularity of the lines S_1S_3 and S_2S_4, the relations (4) result in the quadrilateral $S_1S_2S_3S_4$ being a rhombus. So, the quadrilateral ABCD is parallelogram if and only if the quadrilateral is a rhombus. (fig. 16).

f) From the result of d) and e), we find that the quadrilateral ABCD is a rhombus if and only if it is a square.

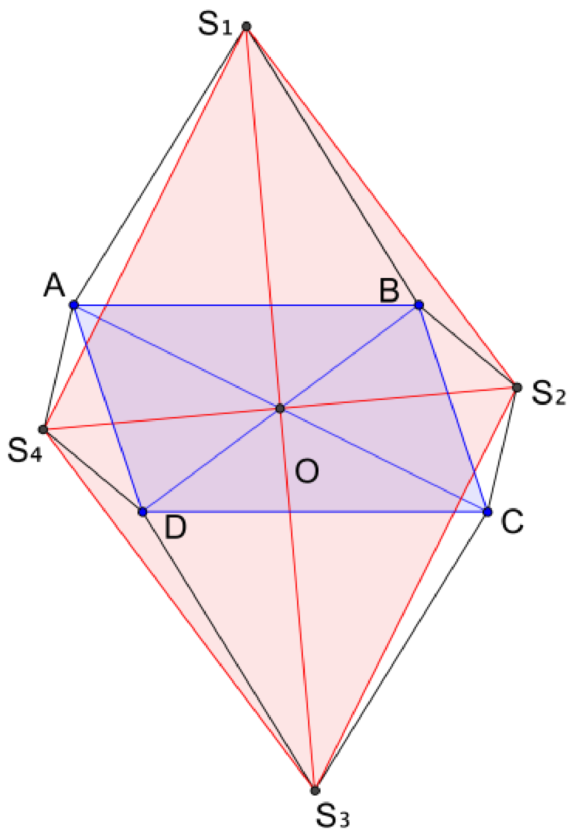


Fig. 16

The theoretical results above have various applications. In fig. 17 we can see the layout of a public garden in the shape of a circle, in which different lines represent different paths. We will consider a rhombus which represents a labyrinth of green bushes. On the sides of the rhombus we can construct right isosceles triangles representing flower beds. Considering point (f), the vertices of those triangles make a square. The diagonals of this square intersect with the diagonals of the rhombus as per point (a). The point of intersection represents a fountain in the centre of the park. The remaining points consist of shops, restaurants and the park's entrance points.

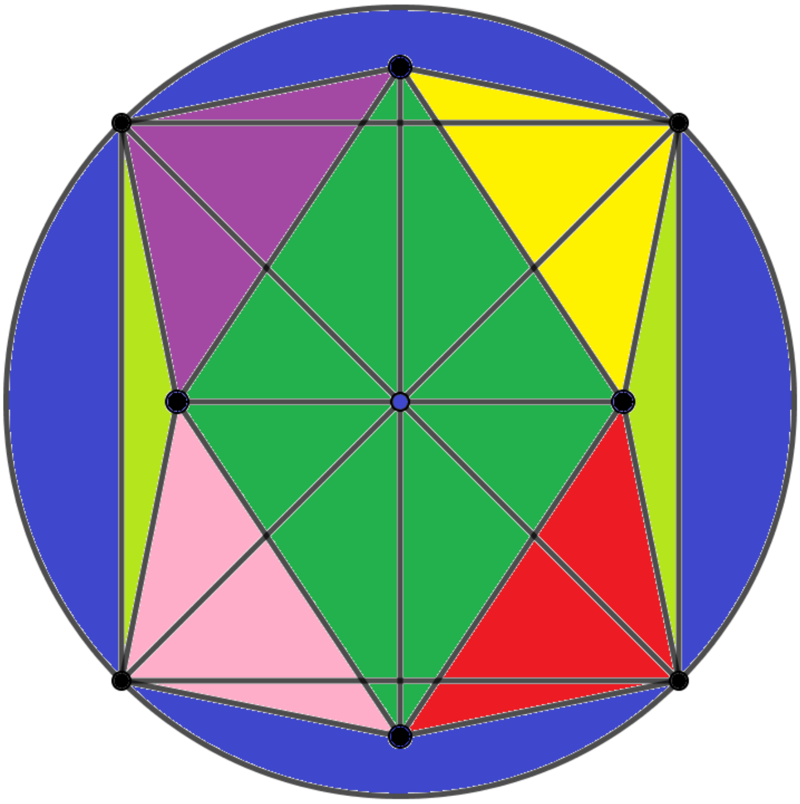


Fig. 17

Notes d'édition

(0) It seems that the subject is to apply some results in geometry to create some flower beds and note to describe, with the help of geometry, existing flower beds.

(1) Understand : the inverse transform is obtained by the direct transform after simple renaming of the triangle vertices.

(2) The application of Pappus' theorem is not clear here. A simple proof using the complex affixes can instead be used. Let m, n, p, m', n' and p' be the affixes of points M, N, P, M', N' and P'. Then the centroid of MNP has for affix

$$(m+n+p)/3 = ((p+m')/2 + (m+n')/2 + (n+p')/2)/3 = (m+n+p)/6 + (m'+n'+p')/6,$$

which leads to

$$(m+n+p)/3 = (m'+n'+p')/3$$

where the right hand side is the affix of the centroid of M'N'P'.

(3) thanks to footnote 1.

(4) The Menelaus theorem states that given a triangle ABC and a transversal line that crosses BC, AC, and AB at points D, E, and F respectively, with D, E, and F distinct from A, B, and C, then AF/FB × BD/DC × CE/EA = −1 (algebraic measures) (Wikipedia).

(5) again thanks to footnote 1.

(6) The A-escribed circle of a triangle ABC is a circle lying outside the triangle, tangent to BC and tangent to the extensions of AB and BC. Every triangle has three distinct excircles, each tangent to one of the triangle's sides (Wikipedia).

(7) Here, A_ABC denotes the area of triangle ABC and p **half the perimeter** of triangle ABC. Hence we can use the identity A_ABC = p × r.

(8) A good question that could be raised is whether this transformation can be inversed.

(9) Indeed, having a=0 can be obtained by translating the figure ; c=1 by rescaling it ; B under the axis by symmetrizing with respect to AC. These operations (translation, rescaling, symmetry) do not change the barycenters nor the angles nor the relative lengths of segments. And D is necessarily above the axis because ABCD is convex.

(10) The segments are orthogonal if, and only if, the ratio of the segments affixes is in iR (not in C-R) : it has to be a pure imaginary number.

(11) m(∡ABC) denotes the measure of angle ABC.

(12) We see from (c) that the points S1, S2, S3, and S4 indeed exist. But we see from (f) that it is not possible to recover ABCD from S1S2S3S4, since there are much more rhombi than squares. Moreover, the proof of (a) gives a construction of S1 (and hence of S2, S3, and S4) unless AOB is isosceles: S1 is the intersection of the angle bisector of AOB with the line bisector of AB. When AOB is isosceles, draw the circumscribed circle of AOB and take S1 as its intersection with the line bisector of AB which is not O.

(13) This is a matter of inscribed angles.