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# The construction of fair 5-sided dice

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## 1 Statement

Is it possible to construct a solid, with 5 faces for example, such that the probability of getting each face is the same,  $1/5$  in the example with 5 faces.

The problem's statement is equivalent to constructing a 5-sided fair dice. Such dice with 4 and 6 faces respectively already exist and are quite popular (for 4 faces there is the tetrahedron and for 6 faces the cube).

In this article we will try to construct a 5-sided dice and from there an  $n$ -sided dice ( $n \geq 4$ ).

## 2 Introduction

To understand the notion of fairness we will first look at some dice that are considered to be fair.

### 2.1 Platonic solids

If a solid has all of its faces congruent, regular polygons, with the same number of faces meeting at each vertex, then, by symmetry, it has the same probability of landing on each of the faces and therefore it is said to be fair.

There are only 5 existing solids that satisfy this property, and they are called the Platonic solids. These solids are the tetrahedron, cube, octahedron, dodecahedron and the icosahedron.

**Definition** (Platonic solid). *A convex polyhedron is a Platonic solid if and only if:*

- (i) *all its faces are congruent convex regular polygons*
- (ii) *all its dihedral angles are congruent*

(iii) the same number of faces meet at each of its vertices

Each Platonic solid can therefore be denoted by a symbol  $\{p, q\}$  where

- (a)  $p$  is the number of edges (or, equivalently, vertices) of each face
- (b)  $q$  is the number of faces (or, equivalently, edges) that meet at each vertex

The symbol  $\{p, q\}$ , called the Schläfli symbol, gives a combinatorial description of the polyhedron[1]. The Schläfli symbols of the five Platonic solids are given in the table below:

Table 1: The Schläfli symbols of the five Platonic solids

Polyhedron	Vertices	Edges	Faces	Schläfli symbol
Tetrahedron	4	6	4	$\{3, 3\}$
Cube	8	12	6	$\{4, 3\}$
Octahedron	6	12	8	$\{3, 4\}$
Dodecahedron	20	30	12	$\{5, 3\}$
Icosahedron	12	30	20	$\{3, 5\}$

This five solids are equally likely to land on any of their faces if rolled at random on any surface. This is true because the Platonic solids are perfectly symmetrical. Any other solid will not be perfectly symmetrical, and therefore we cannot conclude whether or not a given solid is fair or not only by considering its symmetries.

## 2.2 The idea for obtaining an $n$ -sided dice

In an article published in The American Mathematical Monthly, Vol. 96, No. 4. (Apr., 1989) by Persi Diaconis and Joseph B. Keller[2] the two authors defined the notion of 'fairness by symmetry' with which they offered a detailed description of fair dice (solids).

At the end of the article, the authors gave the following example with regard of how to construct a fair polyhedra:

*As an example, let us consider an infinite prism with a regular  $n$ -gon as its cross section. Let us cut it with two planes a distance  $L$  apart perpendicular to its generators, to produce a polyhedron with  $n + 2$  faces. For  $L$  large this solid has very low probability of landing on either of its two ends, whereas, for very small  $L$  it has a high probability of landing on one of them. Therefore by continuity there is some value of  $L$  for which it has the same probability of landing on any one of its  $n + 2$  faces. When  $n$  is odd this yields a fair dice with an odd number of faces.*

In this article we will build a model of a fair 5-sided (and in general  $n$ -sided) dice based on this example.

### 3 Creating the 5-sided dice

**Remark.** *In this section, when referring to dice, we are only referring to dice that have right regular triangular prismatic shape.*

*Also, all the dice are considered to be homogeneous. This means that they have the exact same physical properties at every point.*

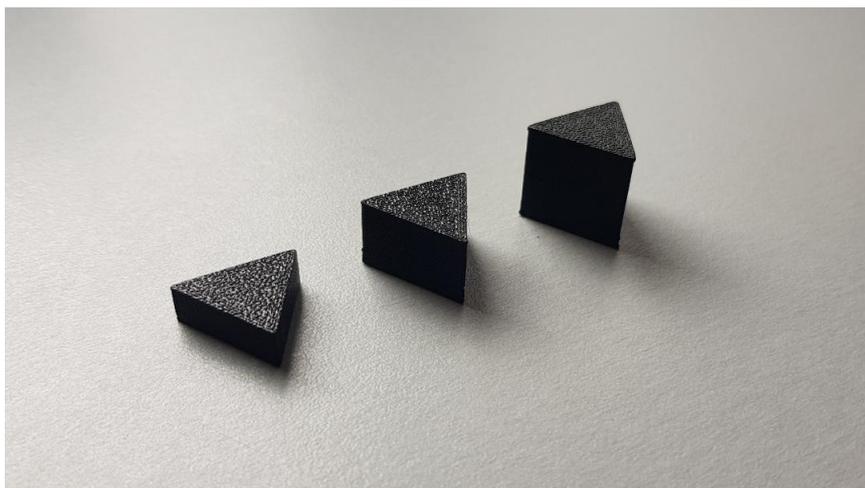


Figure 1: *An example of 3D printed dice. They all have the same base side length, but different heights.*

#### 3.1 Theoretical model

In this section we aim to prove that 5-sided fair dice exist by defining a surjective function.

Afterwards, we will hypothesize a possible shape for the graph of the above-mentioned function.

##### 3.1.1 System of axioms

**Definition** (Similar objects[3][4]). *In Euclidean geometry, two objects are similar if and only if one can be obtained from the other by a scaling (enlarging or reducing), possibly with additional translation, rotation or reflection.*

We based our approach on some intuitive aspects (supported by the experimental part, as we will see later). Not being aware of the existence of any theoretical proofs, we choose to state them as axioms:

**Axiom 1.** *Two similar dice have the same probability of landing on one of the similar faces.*

**Axiom 2.** *For a prismatic dice, if the base area remains constant, the probability of landing on a rectangular face increases with the area of that face. [1]*

**Axiom 3.** *Two dice having the same fixed base side length  $l$  and two comparable heights ( $h_1 \simeq h_2$ ), will also have comparable probabilities of landing on a rectangular face ( $p_{\square 1} \simeq p_{\square 2}$ ).*

### 3.1.2 The model

Let us consider a regular right triangular prism of base side length  $l$  and height  $h$ .

Because of the prism's rotational symmetry, it follows that the two triangular faces will have the same probability of being rolled and so will the three rectangular faces. One rectangular face can be obtained from another by rotating the prism clockwise either  $120^\circ$  or  $240^\circ$ . For the triangular faces, one can be obtained from the other by a  $180^\circ$  rotation.

A fair 5-sided dice will have the same probability of landing on each of the 5 faces, namely:  $p_i = \frac{1}{5}$  for all  $i = 1, \dots, 5$ . Therefore, the probability that a fair 5-sided prism will land on either one of its two triangular faces will be:  $p_{\Delta} = 2 \cdot p_i = \frac{2}{5} = 0.4$ .

We can thus conclude that a 5-sided prismatic dice is fair if and only if

$$p_{\Delta} = 0.4$$

We will denote by  $r$  the ratio of the base side length to the height and we will call  $r$  the prism ratio:

$$r = \frac{l}{h} \quad [2]$$

Given two prisms of the same ratio both of them will have the same probability of landing on a given face, because, in principle, they are the same prism, one of them being just a scaling -up or down- of the other (see Axiom 1). Therefore, there is a unique probability value associated with every  $r$ , which means that any correspondence which associates  $r$  to a certain probability value  $f(r)$  is a function.

Considering this, let us define the following function:

$p_{\mu} : (0, \infty) \rightarrow (0, 1)$  such that

**$p_{\mu}(r)$  is the probability that, after a random roll, a prism of ratio  $r$  will land on either one of the two triangular faces.**

Here  $\mu$  represents a constant that depends on the surface on which the dice is rolled. This constant does not affect the behavior of the function, only affecting the function's outputs.

For example, let us consider two identical dice one of them being rolled on a wooden table and the other on a stack of paper (or an open notebook). A few experimental results easily prove that for the same roll, the dice being rolled on the wooden table bounces back higher than the one rolled on the paper. This happens because wood and paper have different *coefficients of restitution*[5] that is wood and paper have different energy absorption ratios. When a small object falls freely on a surface, part of the object's energy is lost to heat and plastic deformation. The energy loss is higher for paper and lower for wood and thus the object will bounce back higher on wood than on paper.

Therefore, our function depends on the prism through the prism ratio  $r$  and also on the surface on which the dice is rolled through  $\mu$ .

Now, let us state and prove some of the properties of the function  $p_\mu$ :

**Property 1.** *The function  $p_\mu$  is strictly increasing.*

**Property 2.** *The following equalities hold for  $p_\mu$ :*

$$(i) \lim_{r \rightarrow \infty} p_\mu(r) = 1$$

$$(ii) \lim_{r \rightarrow 0} p_\mu(r) = 0$$

**Property 3.** *The function  $p_\mu$  is continuous.*

**Property 4.** *The function  $p_\mu$  is surjective (bijective).*

It now follows from Property 4 that there is a ratio  $r_0$  such that  $p_\mu(r_0) = 0.4$  and thus any prism of such ratio will have the same probability of landing on each of its 5 faces.

Proofs:

**1.** Let us consider two prisms, of ratio  $r_1 = \frac{l_1}{h_1}$  and another of ratio  $r_2 = \frac{l_2}{h_2}$ , such that  $r_2 > r_1$ .

Now, the prism of ratio  $r_2$  is equivalent to a prism of the same ratio which has the base side length equal to  $l_1$ , because this third prism is just a scaling of the second prism with a factor of  $\frac{l_1}{l_2}$  (they are similar- see Axiom 1).

Therefore, this third prism will have a height of  $h_3 = h_2 \cdot \frac{l_1}{l_2}$

Now  $r_1 = \frac{l_1}{h_1}$  and  $r_2 = \frac{l_1}{h_3}$  and since  $r_2 > r_1$  it follows that  $h_1 > h_3$ .

Thus, the triangular areas of both prisms are equal, but the rectangular area is greater for prism 1 than for prism 3. That is:

$$A_{\Delta 1} = A_{\Delta 3}, \quad A_{\square 1} > A_{\square 3}$$

It now follows from Axiom 2 that the third prism will have a smaller probability of landing on the rectangular faces than the first prism.

And since the probability of landing on the rectangular and the probability of landing on the triangular faces are complementary (add up to 1), we have:

$$1 - p_\mu(r_2) < 1 - p_\mu(r_1) \Leftrightarrow p_\mu(r_1) < p_\mu(r_2)$$

Thus  $\forall r_1 < r_2 \in (0, \infty)$  we have  $p_\mu(r_1) < p_\mu(r_2)$  which is equivalent to:

*The function  $p_\mu$  is strictly increasing.*

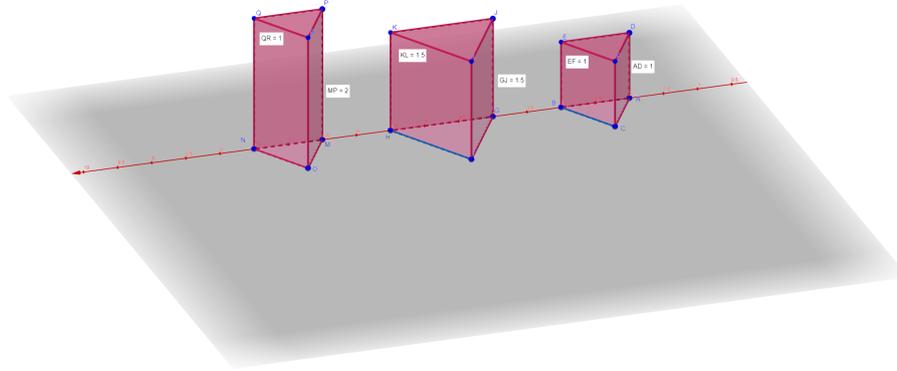


Figure 2: (From left to right) the first prism has a ratio  $r_1 = \frac{1}{2}$ , the second one has a ratio  $r_2 = \frac{1.5}{1.5} = 1$ , and the third one is equivalent to the second one having a ratio  $r_2 = \frac{1}{1} = 1$ . The third prism has the same base side length as the first one, but has a smaller height  $h_3 = 1 < 2 = h_1$ .

**2.** Let us fix a base side length  $l$ , and vary only the height  $h$  to produce a variation of  $r$ .

As the height decreases more and more, the prism is less likely to land on the rectangular face, because the area of the rectangle decreases while the area of the triangle stays the same (see Axiom 2). A prism of height 0 is essentially a 2D triangle, which has probability 1 of being rolled on either one of the 2 faces.

Therefore  $\lim_{r \rightarrow \infty} p_\mu(r) = \lim_{h \rightarrow 0^+} p_\mu\left(\frac{l}{h}\right) = 1$  which is equivalent to:

*The line  $y = 1$  is a horizontal asymptote for the function  $p_\mu$ .*

Moreover, as  $h$  gets larger and larger the area of the rectangle increases while the area of the triangle stays the same. Thus, it follows from Axiom 2 that a prism of infinite height never lands on the triangular faces (in fact it does not even have those faces anymore).

*That is  $\lim_{r \rightarrow 0} p_\mu(r) = \lim_{h \rightarrow \infty} p_\mu\left(\frac{l}{h}\right) = 0$ .*

**3.** [3] As above, let us fix  $l$  and vary  $h$  to produce a variation of  $r$ .

Now, considering two dice of ratios  $R = \frac{l}{H}$  (arbitrarily fixed) and  $r = \frac{l}{h}$  (variable) Axiom 3 is equivalent to: *for every  $\epsilon > 0$ , there exists a  $\delta$  such that*

$$|h - H| < \delta \Rightarrow |p_{\square 1} - p_{\square 2}| < \epsilon$$

and since the probability of landing on the rectangular and the probability of landing on the triangular faces are complementary we get that:

$$p_{\square 1} - p_{\square 2} = (1 - p_\mu(r)) - (1 - p_\mu(R)) = p_\mu(R) - p_\mu(r)$$

Thus, we have:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall H > 0 \text{ with } |h - H| < \delta \Rightarrow |p_\mu(r) - p_\mu(R)| < \epsilon$$

Now, we have only to prove that for all  $\delta > 0$  for which  $|h - H| < \delta$  there exists  $\delta' > 0$  such that  $|r - R| < \delta'$ .

This follows from the equivalences:

$$\begin{aligned}
|h - H| < \delta &\Leftrightarrow H - \delta < h < H + \delta \Leftrightarrow H - \delta < \frac{l}{r} < H + \delta \Leftrightarrow \frac{l}{H + \delta} < r < \frac{l}{H - \delta} \\
&\Leftrightarrow \frac{l}{H + \delta} - \frac{l}{H} < r - R < \frac{l}{H - \delta} - \frac{l}{H} \\
&\Leftrightarrow -\frac{l \cdot \delta}{H(H - \delta)} < -\frac{l \cdot \delta}{H(H + \delta)} < r - R < \frac{l \cdot \delta}{H(H - \delta)} \\
&\Rightarrow |r - R| < \frac{l \cdot \delta}{H(H - \delta)} = \frac{\delta \cdot R}{\left(\frac{l}{R} - \delta\right)} \stackrel{\text{not.}}{=} \delta'
\end{aligned}$$

Therefore, we can conclude that:

$$\forall \epsilon > 0, \exists \delta' > 0 \text{ such that } \forall R > 0 \text{ with } |r - R| < \delta' \Rightarrow |p_\mu(r) - p_\mu(R)| < \epsilon$$

which, by the epsilon-delta definition of a limit is equivalent to  $\lim_{r \rightarrow R} p_\mu(r) = p_\mu(R)$  for all  $R > 0$ . This in fact means that:

*The function  $p_\mu$  is continuous.*

4. Since  $p_\mu$  is continuous and  $\lim_{r \rightarrow \infty} p_\mu(r) = 1, \lim_{r \rightarrow 0} p_\mu(r) = 0$  it follows from the Intermediate Value Theorem that for all  $y \in (0, 1)$  exists  $x \in (0, \infty)$  such that  $p_\mu(x) = y$ .

Thus  $\text{Im } p_\mu \supset (0, 1)$ . Since  $(0, 1)$  is also the codomain of  $p_\mu$ , we have:

$$\text{Im } p_\mu = (0, 1)$$

*That is  $p_\mu$  is a surjection. Moreover, since  $p_\mu$  is strictly increasing, it is an injection, and thus a bijection.*

Therefore, there is a ratio  $r_0$  such that  $p_\mu(r_0) = 0.4$ . Any prism of such ratio will have the same probability of landing on each of its 5 faces.

However, this model is not sufficient to determine the exact ratio  $r_0$  that will yield such a dice.

### 3.1.3 Analyzing the shape of the functions' graph

[4] In this section we will analyze the graph of  $p_\mu$  and search for a function  $m$ , defined by using elementary functions, which has a similar graph.

Even though the function  $m$  is only a 'rough' approximation of  $p_\mu$ , section 3.3.2 will show that for a certain  $\mu$ ,  $m$  behaves very well around the value of  $x = r_0$  which is the ratio of the ideal prismatic dice.

Because the line  $y = 1$  is a horizontal asymptote for  $p_\mu$  and because  $p_\mu(x) \in (0, 1)$  for all  $x \in (0, \infty)$  we can deduce that there exists a value  $a$  such that  $p_\mu$  is concave for all  $x \geq a$ . This is true because if there would be no such value  $a$ , then the function would be convex and thus it would be impossible to have the asymptote  $y = 1$ .

Therefore we have  $p_\mu''(x) < 0$  for all  $x \in (a, \infty)$ , which tells us that the function is concave for the second part.

To determine the convexity of the function for  $x \in (0, a)$  we will analyze the behavior of  $p_\mu$  in the vicinity of  $x = 0$ .

As  $x$  is close to 0 adding a small amount  $dx$  will result in a smaller increase in the outputs of the function than adding the same amount  $dx$  to a value further away from 0.

That is  $p_\mu(x + dx) - p_\mu(x) < p_\mu(y + dx) - p_\mu(y)$  for arbitrary values  $x$  and  $y$  such that  $0 < x < y < a$ .

Dividing this last equation by  $dx$  and applying the limit as  $dx \rightarrow 0$  we get that  $p_\mu'(x) < p_\mu'(y)$ .

It follows that  $p_\mu'$  is strictly increasing for  $x$  values smaller than  $a$ .

Therefore  $p_\mu''(x) > 0$  for  $x \in (0, a)$  which means that  $p_\mu$  is most likely convex for the first part.

Notice that this last result is based on physical intuition and this is the reason why it was stated that  $p_\mu$  is most likely convex. The experimental results of section 3.3.2 will show that this is indeed the case.

We will try to find a possible function  $m$  and graph it. Such a function  $m$  would be convex for  $x \leq a$  and concave for  $x \geq a$ .

For this let us define  $m : (0, \infty) \rightarrow (0, 1)$  such that  $m$  respects all of  $p_\mu$ 's properties.

A function which respects some of the above-mentioned properties (Properties 1, 2 (i), 3, and 4) and which is convex for  $x < 0 = a$  and concave for  $x > 0 = a$ , is the:

Hyperbolic tangent[6]  $\tanh : (-\infty, \infty) \rightarrow (-1, 1)$  defined as  $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ .

However, the range of  $\tanh$  does not correspond to the range of  $m$  and thus it has to be modified. This can be done by adding 1 and then dividing the resulting function by 2.

Let us consider the function  $f : (0, \infty) \rightarrow (0, 1)$  such that

$$f(x) = \frac{\tanh(x - a) + 1}{2}$$

Here, the parameter  $a > 0$  is the one presented earlier and describes the inflection point (i.e. the point at which the  $f$  changes from being convex to being concave)

This function has the appropriate range and satisfies  $\lim_{x \rightarrow \infty} f(x) = 1$  (Property 2 (i)), but it does not satisfy  $\lim_{x \rightarrow 0} f(x) = 0$  (Property 2 (ii)) since

$$\lim_{x \rightarrow 0} f(x) = \frac{\tanh(-a) + 1}{2} > \frac{-1 + 1}{2} > 0$$

This means that we have to change the lower part of our function  $f(x)$ .

For that let us consider the power function  $g : (0, \infty) \rightarrow (0, \infty)$  such that  $g(x) = c \cdot x^b$  where  $b$  and  $c$  are constants which are to be determined from the properties of  $p_\mu$ .

Since  $\lim_{x \rightarrow 0} g(x) = 0$  the function  $g$  has Property 2 (ii), a property which  $f$  did not possess.

Therefore, we can obtain a possible function by defining the piecewise function  $m$  :

$$m(x) = \begin{cases} g(x) & \text{for } x \leq a \\ f(x) & \text{for } x \geq a \end{cases} \quad [5]$$

Now we only need to determine the two constants  $b$  and  $c$  for which  $m$  is continuous and for which the passing from  $f$  to  $g$  is smooth (i.e. the derivative of  $m$  is continuous).

These last conditions can be rewritten as:

$$\begin{cases} f(a) = g(a) \\ f'(a) = g'(a) \end{cases} \Leftrightarrow \begin{cases} \frac{1 + \tanh(a-a)}{2} = c \cdot a^b \\ \frac{1 - \tanh^2(a-a)}{2} = b \cdot c \cdot a^{b-1} \end{cases} \Leftrightarrow \begin{cases} c \cdot a^b = \frac{1}{2} \\ \frac{b}{a} \cdot c \cdot a^b = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} b = a \\ c = \frac{1}{2 \cdot a^a} \end{cases}$$

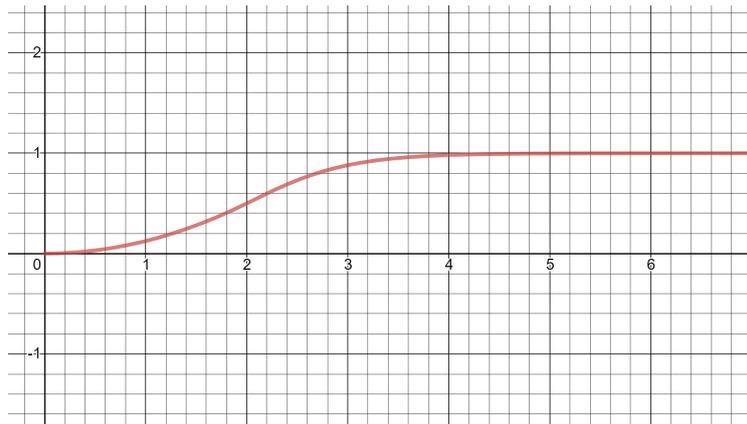


Figure 3: An example showing the function  $m_2$ . The inflection point has coordinates  $I_a(a, \frac{1}{2})$  which in our case is  $I_2(2, \frac{1}{2})$ .

Therefore  $g(x) = \frac{1}{2} \cdot \left(\frac{x}{a}\right)^a$  which gives us the function:

$$m_a : (0, \infty) \rightarrow (0, 1), \quad m_a(x) = \begin{cases} \frac{1}{2} \cdot \left(\frac{x}{a}\right)^a & \text{for } x \leq a \\ \frac{\tanh(x-a) + 1}{2} & \text{for } x \geq a \end{cases}$$

### 3.2 Physical model

This model aims to determine the ratio  $r_0$  for which an **ideal** 5-sided dice is fair. The physics involved in determining the result is applied to an ideal case, and thus the result will not depend on the surface through  $\mu$  as stated in the theoretical model.

### 3.2.1 Boltzmann probability distribution

**Definition** (Boltzmann distribution[8]). [6] *In statistical mechanics, the Boltzmann distribution is a probability distribution that gives the probability of a certain state as function of the state's energy and temperature of the system to which the distribution is applied[8]. The probability distribution is the following[9]:*

$$p_i = \frac{1}{Q} e^{-\frac{\epsilon_i}{k_B T}}$$

Here  $p_i$  represents the probability of state  $i$ ,  $\epsilon_i$  is the energy of state  $i$ ,  $k_B$  is the Boltzmann constant and  $T$  represents the temperature of the system.

The denominator  $Q$  is called the canonical partition function, and is given by the formula:

$$Q = \sum_{i=1}^M e^{-\frac{\epsilon_i}{k_B T}}$$

where  $M$  represents the number of states accessible to the system.

From this definition, it follows that  $\sum_{i=1}^M p_i = 1$ .

This distribution shows that a state with a lower energy will have a higher probability of being achieved than a state with a higher energy [7]. This can be seen by computing the ratio of probabilities for two states,  $i$  and  $j$  :

$$\frac{p_i}{p_j} = e^{\frac{\epsilon_j - \epsilon_i}{k_B T}}$$

### 3.2.2 Predicting the ideal value of $r_0$

Now, considering the dice as a system, there are only two states in which a dice can land after a roll: either on a rectangular face, or on a triangular face.

For a 5-sided dice to be fair, we need to have  $p_i = \frac{1}{5}$  for all  $i = 1, \dots, 5$ . This implies that  $p_{\Delta 1} = p_{\Delta 2} = p_{\square 1} = p_{\square 2} = p_{\square 3} = \frac{1}{5}$  and therefore, the probability of landing on any triangular face must be equal to the probability of landing on any rectangular face.

This in fact means that, for a 5-sided dice to be fair, the ratio of these two probabilities has to be 1:

$$\frac{\frac{1}{5}}{\frac{1}{5}} = 1 = \frac{p_{\Delta}}{p_{\square}} = e^{\frac{\epsilon_{\square} - \epsilon_{\Delta}}{k_B T}} \Leftrightarrow \epsilon_{\Delta} = \epsilon_{\square}$$

Since the dice is at rest on the table it has no kinetic energy, only gravitational potential energy.

**Definition** (Gravitational Potential Energy[10]). *The gravitational potential energy of an object relative to a reference level (the table on which the dice is rolled in our case) is defined as  $U_g = mg\Delta h$  where  $m$  represents the mass of the object,  $g$  is the gravitational acceleration, and  $\Delta h$  represents the height difference between the reference level and the center of mass of the object.*

Therefore, the last equation becomes:

$$U_{g\Delta} = U_{g\Box} \Leftrightarrow \Delta h_{\Delta} = \Delta h_{\Box}$$

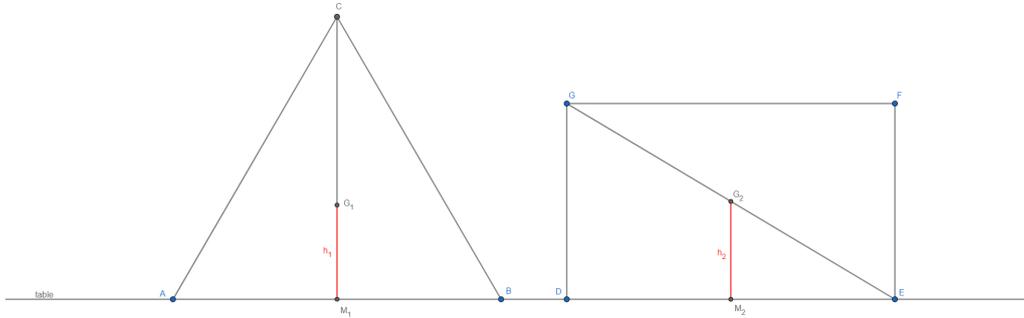


Figure 4: *The side-view of the dice in the two possible states: landed on a rectangular face (left) and on a triangular one (right). The positions of the centers of mass relative to the reference level (table) are denoted by  $h_1$  and  $h_2$ .*

For a homogeneous dice, the center of mass will be located at half of the prism's height, right above the centroid of the base triangle.

Therefore, when the prism will land on the rectangular face we will have:

$$\Delta h_{\Box} = h_1 = G_1 M_1 = \frac{CM_1}{3} = \frac{l\sqrt{3}}{3} = \frac{1}{2} \cdot \frac{l}{\sqrt{3}}$$

and when the prism will land on a triangular face we will have

$$\Delta h_{\Delta} = h_2 = G_2 M_2 = \frac{EF}{2} = \frac{h}{2}$$

$$\Rightarrow \Delta h_{\Delta} = \Delta h_{\Box} \Leftrightarrow \frac{l_0}{\sqrt{3}} = h_0 \Leftrightarrow r_0 = \frac{l_0}{h_0} = \sqrt{3}$$

Thus, the physical model predicts that the ratio  $r_0$  for which a 5-sided dice is fair is  $r_0 = \sqrt{3} \simeq 1.7320$

### 3.3 Experimental model

This model aims to experimentally validate the results of the first two models. All of the measurements were conducted with the dice being rolled on a Plexiglas surface (our  $\mu$  was  $\mu = \mu_{plexiglas}$ ) and thus the final result is dependent on the chosen surface.

We created real-world models of dice with the help of a 3D printer. Unfortunately, the usage of a 3D printer does not guarantee that a dice is perfectly homogeneous, and therefore this could be considered an error source for our experiment.

#### 3.3.1 Methodology

Using a 3D printer, we created 35 dice, each with a different ratio.

All of these dice had the same base side length of  $l = 20mm$ , but different heights, ranging from  $h_{35} = 6.6mm$  to  $h_1 = 18.4mm$ .

For simulating dice throwing we used a collaborative robotic arm with 6 degrees of freedom, a “cobot”, from Universal Robots. A cobot is designed to replicate the movements of the human hand, and work together with humans, automating repetitive, boring tasks, allowing humans to focus on more creative areas.

We also used a photo camera, which took a picture of every roll, allowing us to process it and decide whether the dice landed on a triangular face or on a rectangular one.

The cobot was connected to a Raspberry PI 3 (single board computer), which acted as a communication gateway between the robot and the photo camera. The dice was then placed in a small container with a transparent cap, attached to the cobot arm. To simulate a dice throw, the cobot moved its arm for about 3 seconds after which it stopped the container above camera. Then, the cobot sent a signal to the camera, through the Raspberry PI, with the help of a Python script translate. The camera would take a picture and then the process would begin again.

We did this process 1000 times for each dice, obtaining 1000 dice rolls in less than 2 hours, which allowed us to gather large amounts of data in a short time (Each roll took about 5-6 seconds, which add up to 1 h 20 min - 1 h 40 min per dice).

#### 3.3.2 Results and analysis

In this section, we are going to denote by  $p$  a function that approximates  $p_\mu$  and has all of its properties. Our aim is to find the best possible approximation function  $p$ .

The data we gathered is presented at the end of the document, in table 3 in the Appendix

After considering this scatter plot, we decided to split the function  $p$  into 3 branches (not just 2 as in the theoretical model):

$$p : (0, \infty) \rightarrow (0, 1), \quad p(x) = \begin{cases} f(x) & \text{for } x \leq x_1 \\ g(x) & \text{for } x_1 \leq x \leq x_2 \\ h(x) & \text{for } x \geq x_2 \end{cases}$$

with  $f$  and  $g$  being  $2^{nd}$  degree polynomials and  $h$  being a similar function to the one described in the theoretical model.

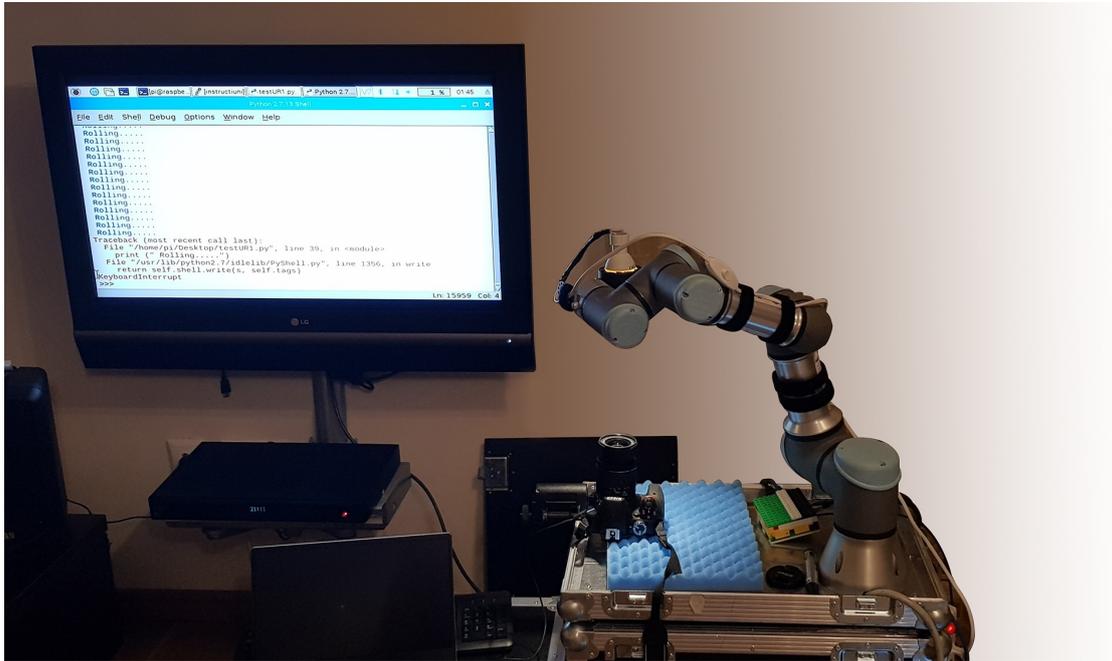


Figure 5: The setup that we used to collect the experimental data: the robotic arm (right), Raspberry PI (encased in the yellow-green lego box), the photo camera and the Python script (rolling on the TV).

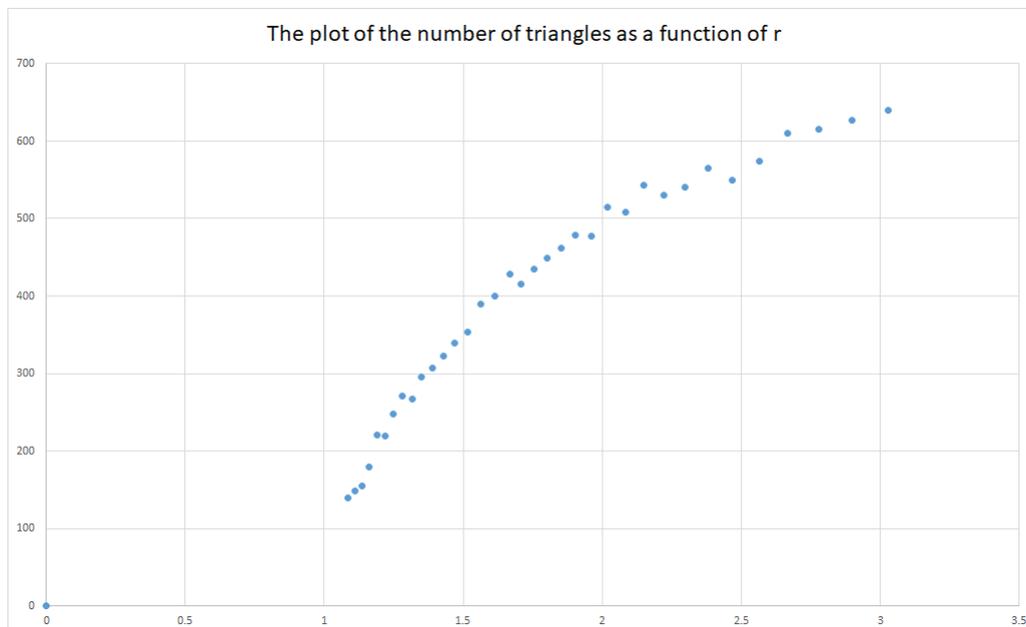


Figure 6: Based on this scatter plot we have determined a possible graph for the function  $p$ .

That is  $h(x) = m \cdot \tanh(rx + q) + n$ , with  $m, n, r$  and  $q$  being constants.

Now, to choose the values  $x_1$  and  $x_2$  we again analyzed the plot to better characterize

the behavior of  $p(x)$ .

Noticing that around the value  $x = 1.35$ , the function changes convexity, we have determined the first value:  $x_1 = 1.35$ .

For the second value, we will choose  $x_2 = 2.5$ , for reasons which will be explained later. Now, the function  $p$  has the following form:

$$p(x) = \begin{cases} ax^2 + bx + c & \text{for } x \leq 1.35 \\ dx^2 + ex + f & \text{for } 1.35 \leq x \leq 2.5 \\ m \cdot \tanh(rx + q) + n & \text{for } x \geq 2.5 \end{cases}$$

Taking into account Property 2 from the theoretical model, we get that:

$$\begin{cases} \lim_{x \rightarrow 0} p(x) = 0 \\ \lim_{x \rightarrow \infty} p(x) = 1 \end{cases} \Leftrightarrow \begin{cases} c = 0 \\ m + n = 1 \end{cases} \Leftrightarrow \begin{cases} c = 0 \\ n = 1 - m \end{cases}$$

Since most of the measurements are in the interval  $[1.35, 2.5]$  we can get an accurate representation of this branch by putting the values in an Excel document and adding a trendline. However, we will not be using the data from table 3, but rather pairs  $(x, y)$  that are more likely to be on the graph of  $g(x)$ .

The chosen  $(x, y)$  pairs are presented in table 2:

Table 2: The values of  $x$  and  $y$  from the 12 chosen pairs

$x$	1.35	1.375	1.4375	1.5	1.625	1.75	1.875	2	2.125	2.25	2.375	2.5
$y$	0.275	0.295	0.33	0.355	0.39	0.44	0.475	0.505	0.53	0.555	0.575	0.595

The values of  $y$  in this table are obtained by determining the correspondent value of  $y$  in Figure 6 and dividing it by 1000 (such that it is in the interval  $[0, 1]$ ).

We graphed this pairs and obtained the following graph and trendline:

Thus we have:

$$g(x) = -0.1492x^2 + 0.8401x - 0.5769$$

We are expressing the constants  $d$ ,  $e$  and  $f$  with 4 decimal points, because from the 5<sup>th</sup> decimal place onward, the decimals are impossible to control (in fact, even the 3<sup>rd</sup> and 4<sup>th</sup> decimals are hard to control, but they are needed for precision). All other constants will be expressed with 4 decimal points as well.

**Remark.** *The reason for which  $x_2$  was chosen to be 2.5 is that for  $x_2 < 2.5$  there would be less data available to accurately graph  $g$ , and for  $x_2 > 2.5$  there would be more data to graph  $g$ , but less data that would be predicted by function  $h$ , and thus  $h$  would be an inaccurate representation.*

Now, let us find the parameter  $m$ . Considering  $r$  and  $q$  fixed, and changing  $m$  we notice that  $m$  determines both the initial "height" of the the function  $h(x)$ , that is

$$h(2.5) = m \cdot \tanh(2.5r + q) + 1 - m$$

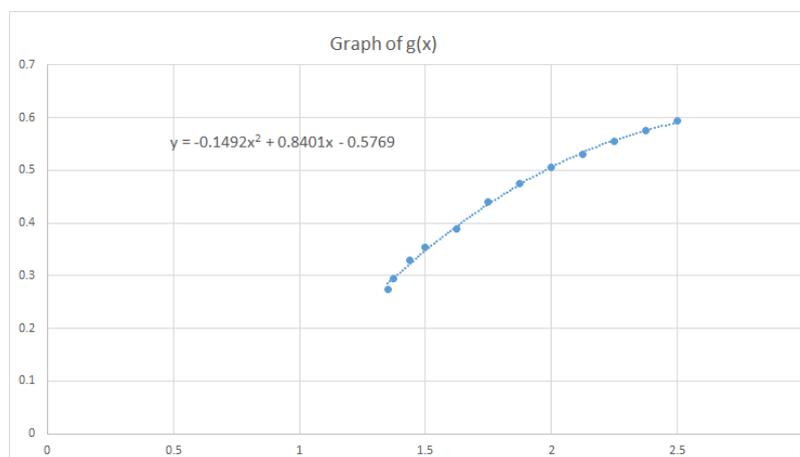


Figure 7: The representation of  $y = g(x)$ ,  $\forall x \in [1.35, 2.5]$ .

and the also curve of the function.

For the function  $h$  to have a similar curve to that of the function  $g$  (this is needed because the graph of  $h$  must prolong the graph of  $g$ ), the value of  $m$  must be around 0.6. For  $m < 0.6$ , the curvature of  $h$  will be smaller than that of  $g$  (blue curves), and for  $m > 0.6$ , it will be larger (green curves).

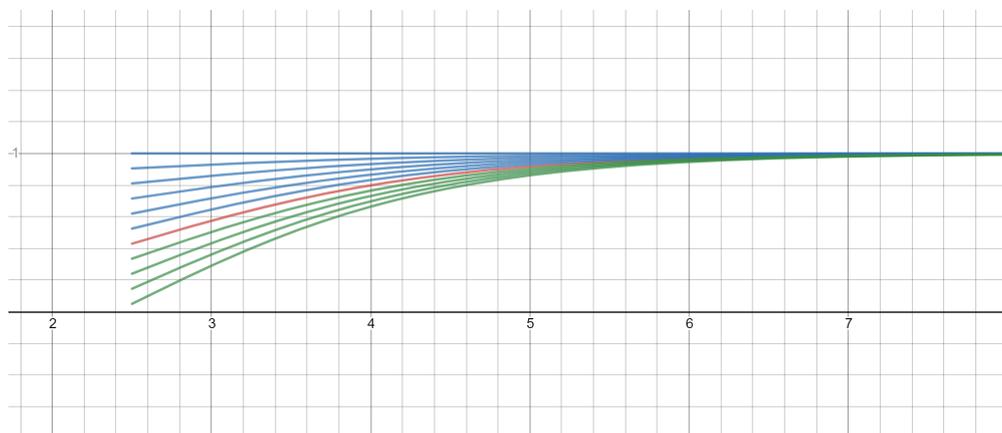


Figure 8: The function  $h(x)$  with the parameters  $r$  and  $q$  fixed ( $r = 0.5$  and  $q = -1.2$ ) for different values of  $m$ :  $m = 0.6$  (red),  $m > 0.6$  (green) and  $m < 0.6$  (blue).

Thus, we can correctly assume that 0.6 is a convenient value for  $m$ .  
Now, the possible function looks like:

$$p(x) = \begin{cases} ax^2 + bx & \text{for } x \leq 1.35 \\ -0.1492x^2 + 0.8401x - 0.5769 & \text{for } 1.35 \leq x \leq 2.5 \\ 0.6 \cdot \tanh(rx + q) + 0.4 & \text{for } x \geq 2.5 \end{cases}$$

The last 4 constants can be determined from the conditions:

- (a) The function  $p$  is continuous.
- (b) The derivative of the function  $p$  is continuous (i.e. the passing from one branch of the graph to another is "smooth").

These can be rewritten as: (1):  $\begin{cases} f(1.35) = g(1.35) \\ f'(1.35) = g'(1.35) \end{cases}$  and (2):  $\begin{cases} h(2.5) = g(2.5) \\ h'(2.5) = g'(2.5) \end{cases}$

$$(1) \Leftrightarrow \begin{cases} 1.35^2 a + 1.35 b = -0.1492 \cdot 1.35^2 + 0.8401 \cdot 1.35 - 0.5769 \\ 2 \cdot 1.35 a + b = -2 \cdot 0.1492 \cdot 1.35 + 0.8401 \end{cases}$$

$$\Leftrightarrow \begin{cases} 1.8225 a + 1.35 b = 0.285318 \\ 2.7 a + b = 0.43726 \end{cases} \Leftrightarrow \begin{cases} a = 0.167343... \\ b = -0.0145(6) \end{cases}$$

$$(2) \Leftrightarrow \begin{cases} 0.6 \cdot \tanh(2.5r + q) + 0.4 = -0.1492 \cdot 2.5^2 + 0.8401 \cdot 2.5 - 0.5769 \\ 0.6 \cdot r \cdot (1 - \tanh^2(2.5r + q)) = -2 \cdot 0.1492 \cdot 2.5 + 0.8401 \end{cases}$$

$$\Leftrightarrow \begin{cases} 0.6 \cdot \tanh(2.5r + q) = 0.19085 \\ 0.6r(1 - \tanh^2(2.5r + q)) = 0.0941 \end{cases} \Leftrightarrow \begin{cases} \tanh(2.5r + q) = 0.31808(3) \\ r = \frac{0.0941}{0.6} \cdot \frac{1}{1 - \tanh^2(2.5r + q)} \end{cases}$$

$$\Leftrightarrow \begin{cases} \tanh(2.5r + q) = 0.31808(3) \\ r = 0.17448745... \end{cases} \Leftrightarrow \begin{cases} q = \tanh^{-1}(0.31808(3)) - 2.5r \\ r = 0.17448745... \end{cases}$$

$$\Leftrightarrow \begin{cases} q = -0.10670538... \\ r = 0.17448745... \end{cases} \quad \text{Thus the 4 constants are: } \begin{cases} a \simeq 0.1673 \\ b \simeq -0.0145 \\ r \simeq 0.1744 \\ q \simeq -0.1067 \end{cases} \quad \text{which give us the}$$

function:

$$p(x) = \begin{cases} 0.1673x^2 - 0.0145x & \text{for } x \leq 1.35 \\ -0.1492x^2 + 0.8401x - 0.5769 & \text{for } 1.35 \leq x \leq 2.5 \\ 0.6 \cdot \tanh(0.1744x - 0.1067) + 0.4 & \text{for } x \geq 2.5 \end{cases}$$

This function is an accurate representation of the experimental data from Table 3 (in fact, for  $1.35 \leq x \leq 2.5$  the function is the most accurate  $2^{nd}$  degree polynomial representation).

Now, by intersecting the function  $p$  with  $y = 0.4$  we can obtain an experimental value for the ratio  $r_0$  for which a 5-sided dice rolled on Plexiglas is fair.

Moreover, since the  $x$  value for which  $p_\mu = 0.4$  is located in the interval  $[1.35, 2.5]$ , it is determined to an even higher degree of accuracy than other values located outside this interval.

Solving the equation  $p_\mu = 0.4$  we get the solution  $r_{0,exp} \simeq 1.6412$  which is around the predicted physical value of  $r_{0,phy} = \sqrt{3} \simeq 1.7320$ .

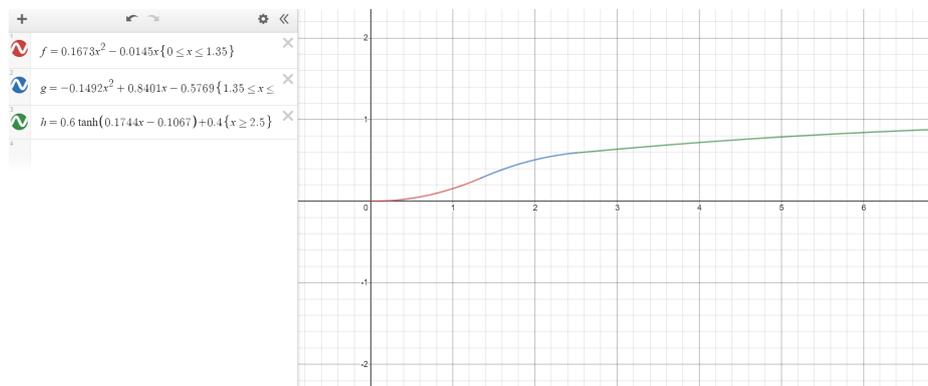


Figure 9: The graph of the approximation function  $p$ .

The relative error,  $\epsilon$ , can be computed by the formula:

$$\epsilon = \frac{\Delta r}{r_{0,phy}} \cdot 100\% = \frac{r_{0,phy} - r_{0,exp}}{r_{0,phy}} \cdot 100\% \simeq 5.24\%$$

Thus, the experimental model validates both the theoretical model and the physical one. The experiment's relative error of 5.24% is well within the acceptable error range. [8]

## 4 Generalization and further research

In this section, we will present an extension of both the theoretical and physical models and prove that it is possible to construct a fair  $n$ -sided dice for any  $n \geq 4$ .

We will also analyze a different model for constructing a  $2n$ -sided fair dice.

### 4.1 Theoretical generalization

For  $n = 4$ , a fair sided dice is the tetrahedron (Platonic solid). No solid with 4 faces can be constructed using a prismatic model, since any prism has a number of faces  $\geq 5$ .

However, for  $n \geq 5$  we will use a prismatic model, just as we used the triangular prism for the 5-sided dice.

Let us consider a regular  $(n-2)$ -gonal right prism (a right prism, having the bases regular  $(n-2)$ -gons and the lateral faces rectangles) of base side length  $l$  and height  $h$ .

Using the same reasoning as for the triangular prism, we can define the ratio of the prism as:

$$r = \frac{l}{h}$$

Now, for a fixed number  $n$ , let us define  $p_{(n,\mu)} : (0, \infty) \rightarrow (0, 1)$  such that:  $p_{(n,\mu)}(r)$  is the probability that, after a random roll, a prism of ratio  $r$  will land on either one of the two bases (regular  $(n-2)$ -gons).

This is, in fact, an extension of the function defined in section 3.1.2 (the function  $p_{(5,\mu)}$  is the same as  $p_\mu$  from section 3.1.2), and using the same methods we can prove that this function has all the properties of its particularization  $p_{(5,\mu)}$ .

A fair  $n$ -sided dice will the same probability of landing on each of the  $n$  faces, namely:  $p_i = \frac{1}{n}$  for all  $i = 1, \dots, n$ . Therefore, the probability that a fair  $n$ -sided prism will land on either one of its two bases will be:  $p_\Delta = 2 \cdot p_i = \frac{2}{n}$ .

Since the function  $p_{(n,\mu)}$  is surjective (see Property 4), it follows that for every  $n$  there is a ratio  $r_{(n,0)}$  such that  $p_{(n,\mu)}(r_{(n,0)}) = \frac{2}{n}$  and thus any prism of such ratio will have the same probability of landing on each of its  $n$  faces.

This ratio will of course be dependent on the surface on which the dice is rolled, dependence described by the surface constant  $\mu$ , just as it was for the 5-sided dice.

In conclusion, a fair  $n$ -sided dice can be constructed, no matter the value of  $n(\geq 4)$ . Yet, for some specific values ( $n = 12$ ,  $n = 20$  for example) there are other fair shapes (dodecahedron for  $n = 12$  and icosahedron for  $n = 20$ ) that are not predicted by our prismatic model.

Our article presents only an example of a family of fair  $n$ -sided dice, not claiming to characterize all of the possible fair  $n$ -sided dice.

## 4.2 Physical generalization

Using the same analysis as in section 3.2 and taking into account that a prismatic  $n$ -sided dice can only be in two possible states after a throw (landing either on a base or on a lateral face) we can deduce that an  $n$ -sided dice is fair if and only if:

$$\Delta h_1 = \Delta h_2$$

where  $\Delta h_1$  and  $\Delta h_2$  represent the height difference between the reference level and the prism's center of mass, in states 1 (landed on a rectangular face) and 2 (landed on a base) respectively.

For the general case, for a regular  $(n - 2)$ -gon we have:

$$\angle AOB = \frac{2\pi}{n - 2} \Rightarrow \angle MOB = \frac{\pi}{n - 2}$$

Then

$$\Delta h_1 = OM = \frac{MB}{\tan(MOB)} = \frac{l}{2} \cdot \frac{1}{\tan\left(\frac{\pi}{n - 2}\right)}$$

and

$$\Delta h_2 = O'M' = \frac{h}{2}$$

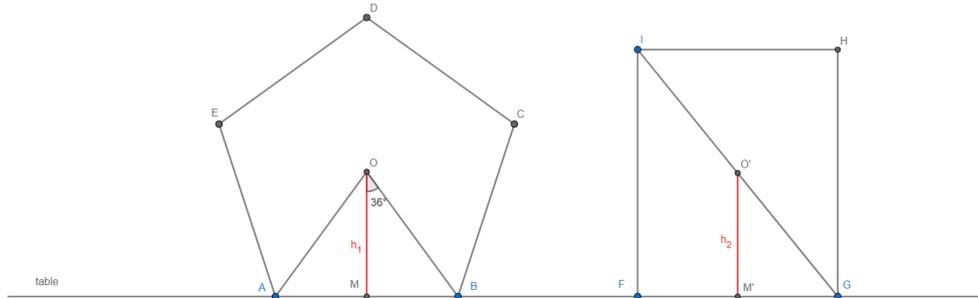


Figure 10: An example for  $n = 7$ . Side-view of the  $n$ -sided dice in its two possible states: landing on a rectangular face (left) and on a regular  $(n - 2)$ -gonal one (right). The positions of the centers of mass relative to the reference level (table) are denoted by  $h_1$  and  $h_2$ .

$$\Delta h_1 = \Delta h_2 \Leftrightarrow r_{(n,0)} = \frac{l_{(n,0)}}{h_{(n,0)}} = \tan\left(\frac{\pi}{n-2}\right)$$

Therefore, the predicted value for which the  $n$ -sided dice is fair is  $r_{(n,0)} = \tan\left(\frac{\pi}{n-2}\right)$ , which is a generalization of the formula found for  $r_0$  in section 3.2.2 (that formula predicted  $r_{(5,0)} = \sqrt{3}$  which is exactly  $\tan\left(\frac{\pi}{3}\right)$ ).

However, this is an idealized result, as it does not take into account the surface on which the dice is rolled. The result is nonetheless a reasonable approximation for real value of  $r_{(n,0)}$ .

### 4.3 The $2n$ -sided fair dice

Creating an odd sided fair dice is much more complicated than creating an even one, because an even sided dice has much more symmetries than an odd sided one.

A possible model for an  $2n$ -sided fair dice is the following:

Two  $n$ -gonal pyramids sharing a common base, and having the same height. This figure could be obtained by gluing together the bases of two identical  $n$ -gonal pyramids.

This object has the same probability of landing on each face because of it's symmetries:

- (a) It is symmetric with respect to the plane of the polygon
- (b) All of it's faces have the same shape and each shares a common edge with only 3 other faces

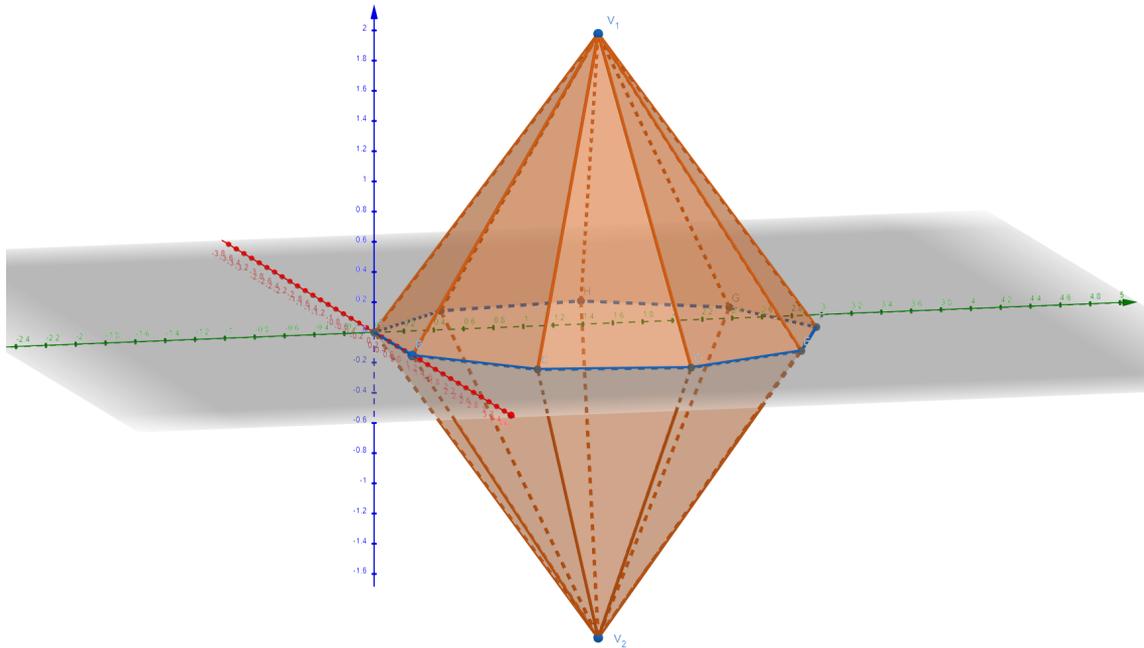


Figure 11: An example of a  $2n$ -sided fair dice for  $n = 9$ . Points  $V_1$  and  $V_2$  are symmetric with respect to the polygon's center.

- (c) Every face meets two other faces at an angle (say  $\alpha$ ) and the third one at another angle (say  $\beta$ ) (for example face  $V_1BC$  meets faces  $V_1AB$  and  $V_1CD$  at the same angle- $\alpha$ - and face  $V_2BC$  at a different angle- $\beta$ )

The only two characteristics that a Platonic solid (section 2.1) has that this shape does not are: the polygons are not necessarily regular (in fact, for  $n \geq 6$  the triangles can never be equilateral) and the dihedral angles are not all congruent.

In fact, for  $n = 4$ , for a suitable height value, we get the octahedron, which is undoubtedly fair.

Because of its many symmetries, this shape is supposedly fair, no matter the value of the height, but a rigorous mathematical proof is still being researched into.

#### 4.4 Further research

We are currently investigating other possible shapes for a fair  $n$ -sided dice, not only the prism.

We are also developing a model that would help us to better understand the fairness of the bipyramidal shape described in section 4.3.

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## Appendix-Experimental Data

Table 3: Experimental data

Dice nr.	Height(h)	Ratio(r)	Nr.triangles	Nr.rectangles	Total
1	18.4	1.087	139	861	1000
2	18	1.111	149	851	1000
3	17.6	1.136	155	845	1000
4	17.2	1.162	180	820	1000
5	16.8	1.190	221	779	1000
6	16.4	1.219	219	781	1000
7	16	1.250	248	752	1000
8	15.6	1.282	271	729	1000
9	15.2	1.315	267	733	1000
10	14.8	1.351	295	705	1000
11	14.4	1.388	307	693	1000
12	14	1.428	322	678	1000
13	13.6	1.470	340	660	1000
14	13.2	1.515	354	646	1000
15	12.8	1.562	390	610	1000
16	12.4	1.613	400	600	1000
17	12	1.666	429	571	1000
18	11.7	1.709	416	584	1000
19	11.4	1.754	435	565	1000
20	11.1	1.801	449	551	1000
21	10.8	1.851	462	538	1000
22	10.5	1.904	479	521	1000
23	10.2	1.960	477	523	1000
24	9.9	2.020	515	485	1000
25	9.6	2.083	509	491	1000
26	9.3	2.150	543	457	1000
27	9	2.222	530	470	1000
28	8.7	2.298	541	459	1000
29	8.4	2.381	565	435	1000
30	8.1	2.469	550	450	1000
31	7.8	2.564	574	426	1000
32	7.5	2.666	610	390	1000
33	7.2	2.777	615	385	1000
34	6.9	2.898	627	373	1000
35	6.6	3.030	640	360	1000

## EDITION NOTES

[1] The additional assumption that the base is a regular polygon is needed. Consider for instance the case in which the bases are hexagons with equal sides, but in which two opposite angles are very small, so that the other four angles are close to 180 degrees. Because of the position of the center of mass, the probability that the prism lands on a face forming the small angle is always zero, independently of the area of that face.

[2] Since, by Axiom 1, the only relevant datum is  $r$ , the whole treatment could have been remarkably simplified by setting  $l = 1$ , or  $h = 1$ . This is actually partially done in the proof of Properties 2 and 3, where  $l$  is fixed.

[3] The following proof could have been rather simpler. Axiom 3 expresses, in an informal way, the continuity of  $p_{\square}$  as a function of  $h$ . Then, since  $p_{\mu}$  is obtained from  $p_{\square}$  by means of elementary operations and, for a fixed value of  $l$ ,  $h$  is a continuous function of  $r$ , we can conclude that  $p_{\mu}$  is continuous as function of  $r$ .

[4] In this subsection, it should be emphasized that there are many possible choices for the components of the function  $m$ . Also, something should be said about why those components are chosen. Why not *arctan* instead of *tanh*, for instance? Also the choice of the point  $a$  as the point where  $m$  changes its concavity seems to be rather arbitrary.

In the second paragraph of this section it is observed that the function  $m$  *behaves very well around the value of  $x = r_0 \dots$* , which seems to confirm the choice of this function. Given that  $m$  is piecewise defined, though, this possible confirmation regards only a ‘piece’ of  $m$ , the one in which  $m(x)$  is near to 0.4 (see Section 3.3.2). In particular, there is no confirmation for the choice of *tanh* as a part of  $m$ .

[5] In the definition of piecewise functions like this, inequality must be strict inequality in one of the two cases. We should have, for instance,  $m(x) = \begin{cases} g(x) & \text{for } x < a \\ f(x) & \text{for } x \geq a \end{cases}$ . The continuity of this function in  $a$  can be proved by showing that  $\lim_{x \rightarrow a^-} g(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$ .

Similarly, we have  $m'(x) = \begin{cases} g'(x) & \text{for } x < a \\ f'(x) & \text{for } x > a \end{cases}$  and, to show that  $m$  is derivable in  $a$ , we have to prove that left and right derivatives of  $m$  in  $a$  exist and coincide.

Everything works in the paper because the functions that we are considering are continuous and have continuous derivatives. But in general continuity and derivability have to be proved as described above.

[6] Boltzmann distribution is generally related to particle systems. This section should have contained some, possibly informal, explanations about why Boltzmann distribution can be used also in the context of the work.

[7] This conclusion can be simply obtained by observing that the function  $x \mapsto e^{-x}$  is decreasing.

[8] As observed in note [4], the experimental model can possibly validate only a part of the theoretical model. Moreover, a natural question here is whether the error is to be viewed as a consequence of statistical fluctuations or as a sign that the model can be improved?