Don't cross the streams

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1 Research topic

The problem we were presented with is the following:

"There's something very important I forgot to tell you ! Don't cross the streams... It would be bad... Try to imagine all your life as you know it stopping instantaneously and every molecule in your body exploding at the speed of light."

Dr. Egon Spengler

Ghostbusters use streams to remove ghosts. As indicated above, it is vital that streams never cross. Let's say that the ten Ghostbusters are chasing ten ghosts.

Is it possible to assign one ghost to every Ghostbusters in such a way that their stream will never cross?

Is it possible to design a computer solution to this problem?

Is it possible to link every Ghostbuster to his ghost without any stream crossing another stream?

2 Results

We proved a much stronger statement than the initial one (the case with ten Ghostbusters and ten ghosts), this being:

Given any natural number n of ghosts and Ghostbusters it is possible to find a correspondence between Ghostbusters and ghosts such that the streams connecting each Ghostbusters to his ghost do not cross, whenever any three of the given points are not collinear.

In the case in which there are three or more collinear points, the correspondence can be made in only some cases, while in others it cannot.

We approached the problem from two very different angles and so, we came up with two distinct solutions, as shown below.

3 The first proof - proving the existence

This solution belongs to Radu Ilieş and Andrei Man.

3.1 Notations

We denote by AB the distance from point A to point B. The symbol $\sum$ stands for summation of the given arguments.

3.2 The initial approach

We started by considering a simpler case: 2 ghosts and 2 Ghostbusters (hunters).

Let G_1 and G_2 be the positions of the ghosts and H_1 and H_2 be the positions of the ghostbusters. If any 3 points aren't on the same line we will have 2 possibilities of connecting the points:

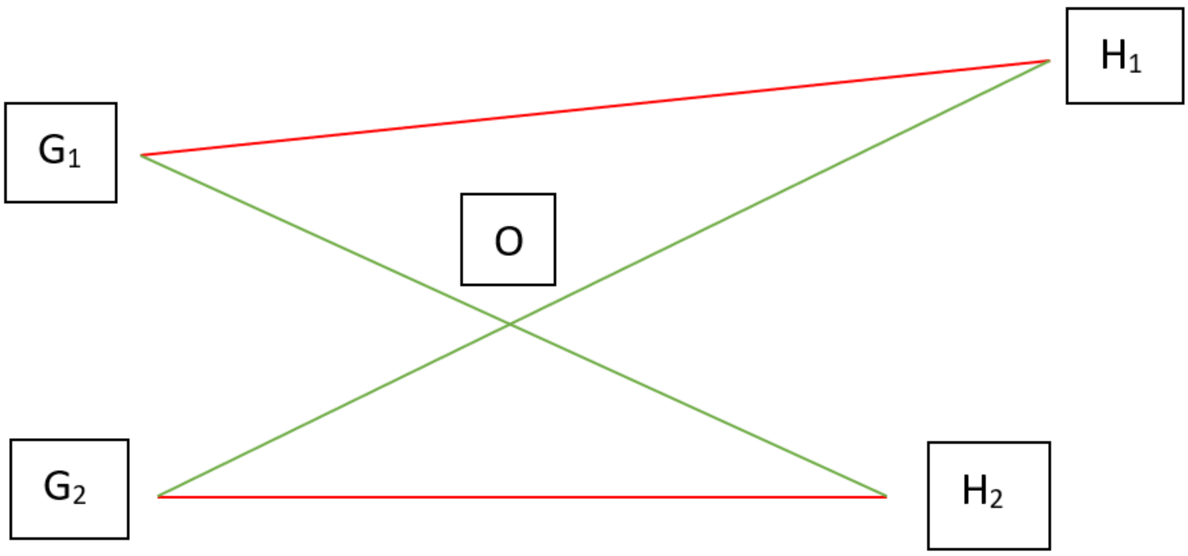

Figure 1: *The particular case $n = 2$*

We can easily find a solution in this case. We have also observed that:

$$H_1G_1 + H_2G_2 < H_1G_2 + H_2G_1$$

and this has led us to the solution for the general case.

3.3 The general case

We will start by considering two sets:

$A = \{H_1, H_2, \dots H_n\}$ and $B = \{G_1, G_2, \dots G_n\}$, which are the positions of the ghostbusters (set A) and the positions of the ghosts (set B) [1].

Now, we consider one-to-one functions (permutations):

$$\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}.$$

For each ghostbuster $H_i \in A$, there is a unique ghost $G_{\sigma(i)} \in B$ and vice-versa.

Let the distance between these two points be $d_i = H_i G_{\sigma(i)}$. For every choice of the function σ, let $S(\sigma)$ be the sum of all those distances,

$$S(\sigma) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} H_i G_{\sigma(i)}.$$

There is a finite number of choices for the function σ (that number is $n!$, and we will prove this later), so it is possible to choose a function reaching the minimum value of $S(\sigma)$.

Let σ_0 be that function (if there are several such functions, we might consider any of them) for which the sum is minimal and the value of the sum for this function is

$$S(\sigma_0) = \sum_{i=1}^{n} H_i G_{\sigma_0(i)}.$$

Now let's assume that for this situation (in which the function σ_0 matches the points between the two sets A and B) there are two segments $(H_i G_{\sigma_0(i)})$ and $(H_j G_{\sigma_0(j)})$ which intersect, where $i, j \in \mathbb{N}$ and $1 \leq i < j \leq n$.

Let $H_i G_{\sigma_0(i)} \cap H_j G_{\sigma_0(j)} = \{O\}$.

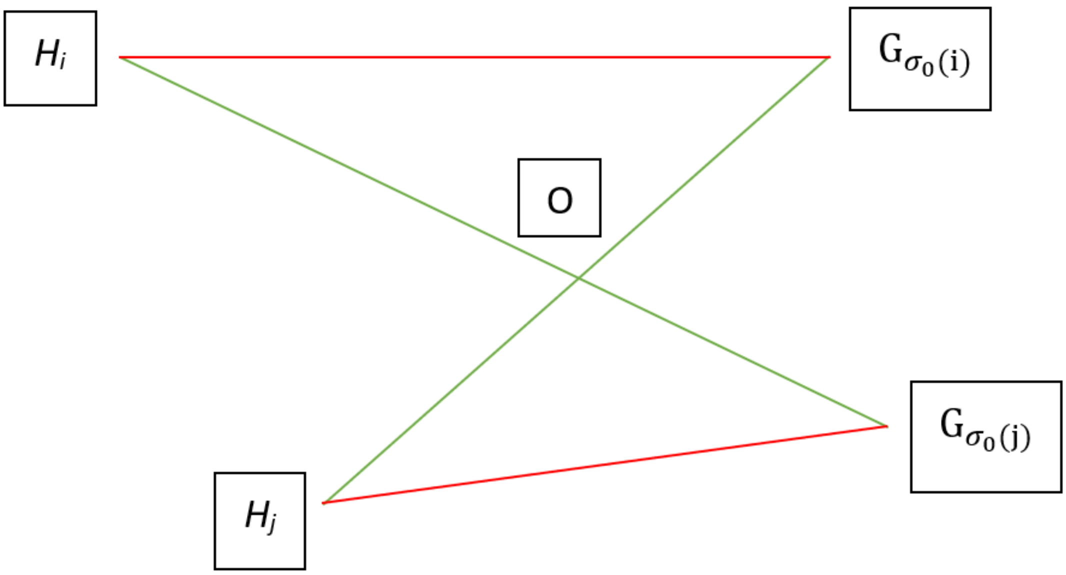

Figure 2: *The application of the triangle inequality in the general case*

Now we will apply the triangle inequality in the triangles $\triangle H_i O G_{\sigma_0(j)}$ and $\triangle H_j O G_{\sigma_0(i)}$:

$$OH_i + OG_{\sigma_0(j)} > H_i G_{\sigma_0(j)} \tag{1}$$
$$OH_j + OG_{\sigma_0(i)} > H_j G_{\sigma_0(i)}. \tag{2}$$

By adding these two inequalities we get:

$$OH_i + OG_{\sigma_0(i)} + OH_j + OG_{\sigma_0(j)} > H_iG_{\sigma_0(j)} + H_jG_{\sigma_0(i)} \tag{3}$$
$$H_iG_{\sigma_0(i)} + H_jG_{\sigma_0(j)} > H_iG_{\sigma_0(j)} + H_jG_{\sigma_0(i)} \tag{4}$$

Therefore there exists a sum whose value is less than $S(\sigma_0)$ [2], contradiction with the minimality of $S(\sigma_0)$.

Thus, none of the segments will have a common point in the situation given by $\boldsymbol{\sigma_0}$.

Moreover, setting the coordinates of the points we can compute all these sums and find not only the value of $\boldsymbol{\sigma_0}$, but also the way we should match the points as the segments do not intersect.

Now we will show that we have $n!$ choices for a one-to-one function:
$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ where n is a natural number greater than 1.
$f(1)$ can be chosen in n ways.
$f(2)$ can be chosen in $n-1$ ways (because it is a one-to-one function and the value of $f(1)$ is already taken).
...
$f(n)$ can be chosen in 1 way.
By the principle of multiplication, there will be $n \cdot (n-1) \cdot \ldots \cdot 1 = n!$ such functions.

4 The second proof - finding the correspondence

This solution was given by Mihnea Leonte.

4.1 Notations

We denote by

(AB)	the open line segment determined by A and B
$[AB]$	the closed line segment determined by A and B
$(AB$	the open half-line with initial point A containing B
$\mathrm{Int}(\widehat{AOB})$	the set of the interior points of the angle $\widehat{AOB}$ [3]
$\mathrm{Ext}(\widehat{AOB})$	the set of the exterior points of the angle $\widehat{AOB}$
$\mathrm{Int}(\triangle ABC)$	the set of the interior points of the triangle ABC

Remarks :
$M \in \mathrm{Int}(\widehat{AOB}) \Leftrightarrow m(\widehat{AOM}) + m(\widehat{BOM}) = m(\widehat{AOB})$ and $m(\widehat{AOM}) \neq 0 \neq m(\widehat{BOM})$
$\mathrm{Int}(\triangle ABC) = \mathrm{Int}(\widehat{ABC}) \cap \mathrm{Int}(\widehat{ACB}) \cap \mathrm{Int}(\widehat{BAC})$

4.2 Explaining the idea

We started with a few cases with small numbers of characters(5-6 points of each type) and we tried to find a convenient correspondence between ghosts and ghostbusters. After determining such a correspondence in each case we searched for a common property and found out that in every configuration there was a ghost-hunter line which splits the plane into two half-planes such that each half-plane contains the same number of ghosts and ghostbusters.

We tried to determine this line by choosing the point (O) with the minimum x-coordinate and then numbering the other points in a way that indicates how many characters are contained in a specific angle formed with this point.

This method is very similar to some methods used to determine the convex hull of a set of points and so we did some research on this topic.

In mathematics, the convex hull of a set X of points in the Euclidean plane is the smallest convex set that contains X.
For instance, when X is a bounded subset of the plane, the convex hull may be visualized as the shape enclosed by a rubber band stretched around X.
(https://en.wikipedia.org/wiki/Convex_hull - Monday June 3rd 2019, 15:29)

Our solution has some common ides with two of the algorithms used to determine the convex hull of a finite set of points (*Graham scan algorithm* and *Divide and conquer algorithm*).
The solution we found is the following:

4.3 The proof

We work in the case when there are no three characters (ghosts or Ghostbusters) disposed in a straight line.

Let us consider a Cartesian coordinate system in the plane and let

$$\mathcal{V}_n = \{V_i(x_i, y_i) \mid i = 1, \ldots, n\},\ \mathcal{F}_n = \{F_i(z_i, t_i) \mid i = 1, \ldots, n\}$$

be the set of hunters (Ghostbusters) and the set of ghosts, respectively. Obviously, any three points from $\mathcal{V}_n \cup \mathcal{F}_n$ are not collinear.

Let us consider a bijective function $g : \mathcal{V}_n \to \mathcal{F}_n$ with

$$g(V_i) = F_{j_i}\ ,\ \ i = 1, \ldots, n\ .$$

It means that $\{j_1, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

To solve the problem we will prove that one can construct a function g having the property

$$\mathcal{P}:\ \ \textit{Any two of the line segments } (V_1F_{j_1}), \ldots, (V_nF_{j_n}) \textit{ do not intersect.}$$

We prove by way of induction on n the following statement

$$P(n):\ \text{Given } n \in \mathbb{N}^*,\ \text{one can construct a bijection } g \text{ having the property } \mathcal{P}.$$

For $n = 1$ the statement is obviously true since the only bijection – defined by $g(V_1) = F_1$ – has the property $\mathcal{P}$.

Let us show that $P(2)$ holds. In this case, $\mathcal{V}_2 = \{V_1, V_2\},\ \mathcal{F}_2 = \{F_1, F_2\}$ and one can define $g : \mathcal{V}_2 \to \mathcal{F}_2$ in two ways

$$\begin{cases} g_1(V_1) = F_1 \\ g_1(V_2) = F_2 \end{cases} \quad \text{or} \quad \begin{cases} g_2(V_1) = F_2 \\ g_2(V_2) = F_1 \end{cases}$$

Let us assume by way of contradiction that none of these functions has the property $\mathcal{P}$. Then

$$(V_1F_1) \cap (V_2F_2) \neq \varnothing \text{ and } (V_2F_1) \cap (V_1F_2) \neq \varnothing.$$

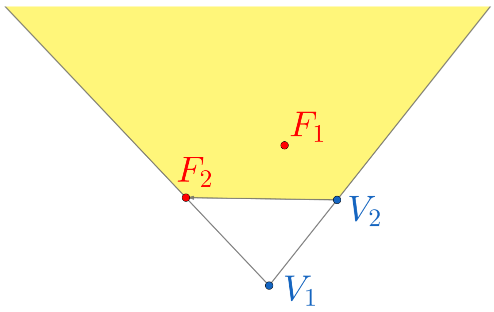

Figure 3: *The part of the plane containing F_1 provided that $(V_1F_1) \cap (V_2F_2) \neq \varnothing$*

The segments (V_1F_1) and (V_2F_2) intersect if and only if

$$F_1 \in \mathrm{Int}(\widehat{V_2V_1F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2).$$

We work with open segments since there are no three collinear points in $\mathcal{V}_2 \cup \mathcal{F}_2$. E.g., if F_1 were on $[V_2F_2]$, on $(V_1F_2 \backslash (V_1F_2]$, or on $(V_1V_2 \backslash (V_1V_2]$, then this non-collinearity condition would not hold.

Analogously, for the segments (V_2F_1) and (V_1F_2) to intersect one must have

$$F_1 \in \mathrm{Int}(\widehat{V_1V_2F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2).$$

Therefore

$$\begin{cases} (V_1F_1) \cap (V_2F_2) \neq \varnothing \\ (V_2F_1) \cap (V_1F_2) \neq \varnothing \end{cases} \Leftrightarrow \begin{cases} F_1 \in \mathrm{Int}(\widehat{V_2V_1F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2) \\ F_1 \in \mathrm{Int}(\widehat{V_1V_2F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2) \end{cases}$$

Since $\mathrm{Int}(\widehat{V_2V_1F_2}) \cap \mathrm{Int}(\widehat{V_1V_2F_2}) = \mathrm{Int}(\triangle V_2V_1F_2)$ [4], the above condition becomes

$$F_1 \in \Big(\mathrm{Int}(\widehat{V_2V_1F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2)\Big) \cap \Big(\mathrm{Int}(\widehat{V_1V_2F_2}) \backslash \mathrm{Int}(\triangle V_2V_1F_2)\Big) = \varnothing,$$

which is impossible.

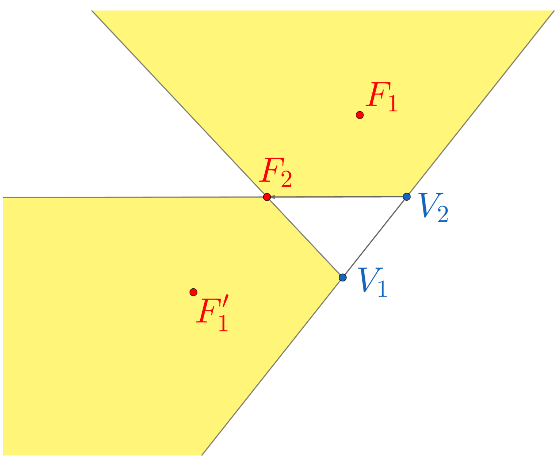

Figure 4: *The parts of the plane where F_1 must lie (in the same time) in order to have $(V_1F_1) \cap (V_2F_2) \neq \varnothing$ and $(V_1F_2) \cap (V_2F_1) \neq \varnothing$. One can easily notice that these parts have no common points (since we are working with open segments, F_2 is not a common element for the two colored plane sections).*

This contradiction shows that our assumption must be wrong, thus either g_1, or g_2 has the property $\mathcal{P}$, and $P(2)$ is true.

Assume that the statements $P(1)$, $P(2)$, …,$P(n)$ are proved and let us show that $P(n+1)$ holds. For this, we prove that for any points $V_1, V_2, \ldots, V_{n+1}, F_1, F_2, \ldots, F_{n+1}$ such that any three of them are not collinear,

there exist some $i, j \in \{1, 2, \ldots, n+1\}$ such that the line V_iF_j determines two open half-planes S_1 and S_2 such that for some $k \in \{0, 1, \ldots, n\}$, S_2 contains exactly $n-k$ hunters and exactly $n-k$ ghosts.

Of course, this means that all the other k ghosts and k hunters are in S_1.

The case $k = 0$ corresponds to the situation when all the ghosts and hunters, except for V_i and F_j are in S_2, hence we are in the situation of $P(n)$, which is true, and the case $k = n$ is similar. If $1 \leq k < n$, all the (open) line segments connecting points from S_i are in S_i $(i = 1, 2)$, hence the statement $P(n+1)$ is true since $P(n-k)$ and $P(k)$ are true. Indeed, the hypothesis of the induction step provides us with the bijections

$$g' : \mathcal{V}_{n+1} \cap S_2 \to \mathcal{F}_{n+1} \cap S_2 \text{ and } g'' : \mathcal{V}_{n+1} \cap S_1 \to \mathcal{F}_{n+1} \cap S_1$$

which satisfy $\mathcal{P}$ and, consequently, taking $j_i = j$, the function

$$g : \mathcal{V}_{n+1} \to \mathcal{F}_{n+1},\ g(V_m) = \begin{cases} F_{j_i} = F_j, & \text{if } V_m = V_i \\ g'(V_m), & \text{if } V_m \in \mathcal{V}_{n+1} \cap S_2 \\ g''(V_m), & \text{if } V_m \in \mathcal{V}_{n+1} \cap S_1 \end{cases}$$

is a bijection which satisfies $\mathcal{P}$.

Thus, finding a line V_iF_j which determines the half-planes S_1 and S_2 satisfying the above conditions completes the induction step and ends the solution.

Next, we will explain how to construct the above-mentioned line V_iF_j. The algorithm we implemented is based on this construction.

First, let us translate the initial Cartesian coordinates system such that the point from $\mathcal{V}_{n+1} \cup \mathcal{F}_{n+1}$ with the minimum x-coordinate $(\min(\min\limits_{i=1,\ldots,n} x_i; \min\limits_{i=1,\ldots,n} z_i))$ becomes the new origin O. If there are two such points in $\mathcal{V}_{n+1} \cup \mathcal{F}_{n+1}$, one can rotate the initial system with an appropriate angle α in order two obtain only one point fulfilling this condition. With no loss of generality, one can consider that this point is a hunter. If it is a ghost, one can get the right construction by simply permuting the letters V and F.

Let this hunter be V_o and let us consider the half-lines $(V_oF_1, (V_oF_2, \ldots, (V_oF_{n+1}$. We rearrange the points $F_1, F_2, \ldots, F_{n+1}$ by the measure of the angle these half-lines form with $(OY = (V_oY$ as follows

$$0 < m(\widehat{F_{a_0}OY}) < m(\widehat{F_{a_1}OY}) < \ldots < m(\widehat{F_{a_n}OY}) < 180^\circ$$

where $\{a_0, a_1, \ldots, a_n\} = \{1, 2, \ldots, n+1\}$.

For any $k = 0, \ldots, n$, we consider the pairs of nonnegative integers (v_k, f_k), where v_k denotes the number of all the hunters in $\text{Int}(\widehat{F_{a_k}OY})$ and f_k denotes the number of all the ghosts in $\text{Int}(\widehat{F_{a_k}OY})$.

From the way we arranged the points $F_{a_0}, F_{a_1}, \ldots, F_{a_n}$ it follows that

$$f_k = k,\ \forall k = 1, \ldots, n$$

since all the ghosts which determine interior points of $\widehat{F_{a_k}OY}$ are $F_{a_0}, \ldots, F_{a_{k-1}}$ and $f_0 = 0$ since $\text{Int}(\widehat{F_{a_0}OY})$ contains no ghost. Hence

$$(v_k, f_k) = (v_k, k),\ \forall k = 0, \ldots, n.$$

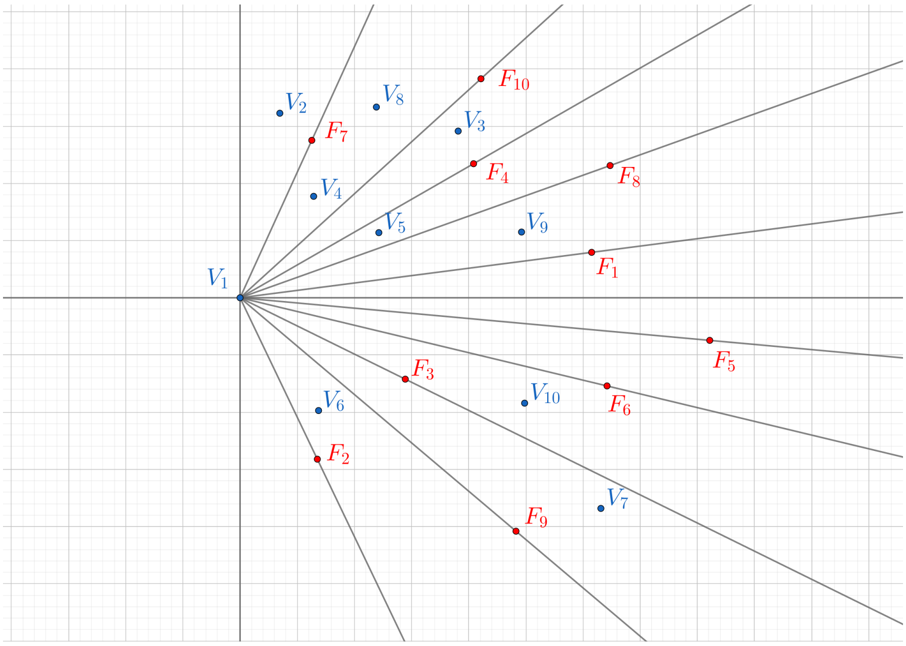

Figure 5: *An example of convenient choice of the coordinates system. Here,* $V_o = V_1 = O$ *and* $a_1 = 7$, $a_2 = 10$, $a_3 = 4$, $a_4 = 8$, $a_5 = 1$, $a_6 = 5$, $a_7 = 6$, $a_8 = 3$, $a_9 = 9$ *and* $a_{10} = 2$

We prove by way of contradiction that for some $j \in \{a_0, \dots, a_n\}$ the line V_oF_j determines two open half-planes such that in each of them we have the same number of ghosts and hunters. Assume that such a j does not exist. This means that

$$\nexists k \in \{0, 1, \dots, n\} \text{ such that } v_k = f_k$$

or, equivalently,

$$\forall k \in \{0, 1, \dots, n\}, \;\; v_k \neq f_k \Leftrightarrow \forall k \in \{0, 1, \dots, n\}, \;\; v_k \neq k.$$

Since $\text{Int}(\widehat{F_{a_k}OY}) \subset \text{Int}(\widehat{F_{a_{k+1}}OY})$, the number of hunters in $\text{Int}(\widehat{F_{a_{k+1}}OY})$ is at least the number of hunters in $\text{Int}(\widehat{F_{a_k}OY})$, hence $v_{k+1} \geq v_k$. Also, $v_k \geq 0$ for any $k = 1, \dots, n$. Thus

$$v_n \geq ... \geq v_1 \geq v_0 \geq 0.$$

But

$$\begin{cases} v_0 \neq 0 \\ v_0 \geq 0 \end{cases} \xRightarrow{v_0 \in \mathbb{N}} v_0 \geq 1 \Rightarrow v_1 \geq 1.$$

Then

$$\begin{cases} v_1 \neq 1 \\ v_1 \geq 1 \end{cases} \xRightarrow{v_1 \in \mathbb{N}} v_1 \geq 2 \Rightarrow v_2 \geq 2,$$

and so on. Finally, we have

$$\begin{cases} v_{n-1} \neq n-1 \\ v_{n-1} \geq n-1 \end{cases} \xRightarrow{v_{n-1} \in \mathbb{N}} v_{n-1} \geq n \Rightarrow v_n \geq n$$

and

$$\begin{cases} v_n \neq n \\ v_n \geq n \end{cases} \xRightarrow{v_n \in \mathbb{N}} v_n \geq n+1,$$

which is impossible, because the maximum possible value for v_n is n (the plane was supposed to contain $n+1$ hunters and none of the counts $v_0, v_1, \ldots, v_n$ includes V_o).

This contradiction shows that our assumption must be wrong, so there exists some k such that $v_k = k = f_k$. The line $V_oF_{a_k}$ is the line V_iF_j we wanted to construct.

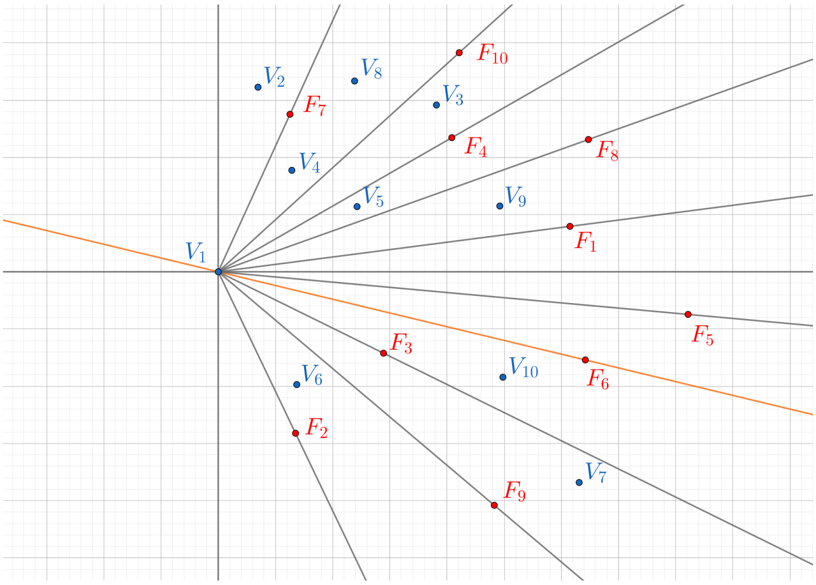

Figure 6: *In this example, V_1F_6 is the line we are searching for. After drawing this line, the corresponding statement $P(10)$ split into $P(3)$ (the lower half-plane determined by V_1F_6) and $P(6)$ (the upper half-plane determined by V_1F_6)*

Remark: 1. For defining the bijection g, we have $g(V_o) = F_{a_k}$. As we saw, $v_k = f_k = k$, and all the other values of g can be determined by successively applying the above algorithm to the remaining k hunter-ghost pairs above $V_oF_{a_k}$ and to the remaining $n-k$ hunter-ghost pairs below $V_oF_{a_k}$.

2. Another convenient way of choosing a Cartesian coordinates system would have been to translate the initial one so that the point from $\mathcal{V}_{n+1} \cup \mathcal{F}_{n+1}$ with the minimum y-coordinate $(\min(\min_{i=1,\ldots,n} y_i; \min_{i=1,\ldots,n} t_i))$ becomes the new origin O. In this case we would rearrange the points by the measure of the angle that the half-lines $(V_oF_1, (V_oF_2, \ldots, (V_oF_{n+1}$ make with $(OX = (V_oX$ (which is equivalent to rearranging the points by the value of the slope of these half-lines).

5 Conclusions and further research

The initial problem asks the readers if a non-crossing ghost-hunter correspondence can be found in any case. As stated in the beginning of this article, the case when 3 or more characters are collinear can only sometimes be solved. It would seem that in this case, even the position of the $2n$ points matters, and since our proofs are based on the assumption that no 3 characters are collinear, they do not include this case. We will continue to study the problem in order to find a solution for this particular case.

A possible generalization of this topic is to turn it into a problem of solid geometry. Having $2n$ points in space can one find a correspondence which satisfies the conditions of the initial problem ? A starting idea would be trying to project all of the points into a plane such that no three characters are collinear. If such a plane could be found, then this would solve the problem, since the 2D version is already solved. The hardest part is determining a way to find this plane, if one even exists.

This generalization is still being researched and thus we cannot give a conclusive answer yet.

Editing Notes

[1] It is assumed that any three of the points are not collinear.

[2] Namely the sum $S(\sigma)$ corresponding to the function σ where the values $\sigma_0(i)$ and $\sigma_0(j)$ are exchanged ($\sigma(i) = \sigma_0(j)$, $\sigma(j) = \sigma_0(i)$ and $\sigma(k) = \sigma_0(k)$ if $k \neq i$ and $k \neq j$).

[3] The "angle $\widehat{AOB}$" denotes also the whole region of the plane lying between the half lines $(OA$ and $(OB$; the side is chosen such that its measure $m(\widehat{AOB})$ lies between $0°$ and $180°$.

[4] See the Remark section 4.1. In fact any intersection of two regions among $\mathrm{Int}(\widehat{ABC})$, $\mathrm{Int}(\widehat{ACB})$ and $\mathrm{Int}(\widehat{BAC})$ is equal to $\mathrm{Int}(\triangle ABC)$.