DECOMPOSING INTEGERS

Year 2020 – 2021

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"God created the integers, all else is the work of man." Leopold Kronecker

The Problem. What are **the integers** that can be written as $\boldsymbol{x^2+ay^2}$, where $\boldsymbol{a} \in \mathbf{Z}$ is fixed?

Remark. Finding $n \in \boldsymbol{Z}$ that can be written as x^2+ay^2 is like looking for $n \in \mathbf{Z}$ for which the equation $x^2+ay^2=n$ has solutions $(x,y) \in \mathbf{Z}\times\mathbf{Z}$. It will be noticed that some equations have no solutions, while others have a finite number of solutions and others have an infinite number of solutions.

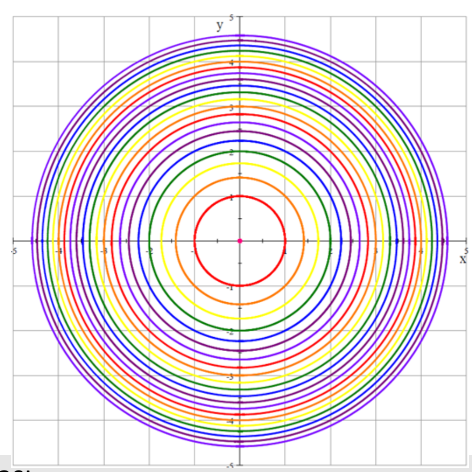
Using the Scientific WorkPlace Program for graphing, we will analyze the important cases $a=1$, $a=-1$, $a=2$, $a=-2$, discovering the solvability of equations by graphical search on circles, ellipses, lines and hyperboles. For *a*=0, we will also mention the graphical answer.

We will present some general results demonstrated in number theory, which are found in the works in the bibliography, both for $a \in \{1,2,-1,-2,0\}$ and for other values. We will also mention open problems.

A computer program could be written to generate, for each fixed $a \in \mathbf{Z}$, the numbers $n=x^2+ay^2 \in \mathbf{Z}$, giving values $(x,y) \in \mathbf{N}\times\mathbf{N}$ and then ordering the generated integers. Having an infinite number of integers, the program must be stopped running imposing an upper generation limit.

I. $a>0$ ($a=1,2,3,5,7$, other cases)

Remark. Since $x^2+ay^2 \geq 0$, $\forall (x,y) \in \mathbf{Z}\times\mathbf{Z} \Rightarrow n \in \mathbf{Z}, n<0$ cannot be written as x^2+ay^2, $(x,y) \in \mathbf{Z}\times\mathbf{Z}$.
We look for $n \in \mathbf{Z}, n \geq 0$ for which there is $(x,y) \in \mathbf{Z}\times\mathbf{Z}$ such that $n=x^2+ay^2$.



QUESTION 1. $\boldsymbol{a=1}$. What are integers $n \in \mathbf{Z}, n \geq 0$ that can be written as $\boldsymbol{x^2+y^2}$?

Answer 1.1. We partially researched, by **graphical analysis** and algebraic verification, which of the numbers $n \in \{0,1,2,\dots,21\}$ can be written as x^2+y^2.

1) $n=0$: We are looking if the circle that becomes a double point, with the equation

$$x^2+y^2=0 \Leftrightarrow (x,y)=(0,0),$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. We find: $n = 0 = 0^2 + 0^2$. (1)

2) $n \in \mathbf{Z}, n \geq 1$: We are looking if the circle with center $(0,0)$ and radius $\sqrt{n}$, with the equation

$$x^2 + y^2 = n \Leftrightarrow x^2 + y^2 = (\sqrt{n})^2$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. By searching for $n \in \{1,2,\dots,21\}$, i.e. looking for the colored circles passing through integer coordinate points, we find *all possible writing solutions*:

red: $1 = 1^2 + 0^2 = 0^2 + 1^2 = (-1)^2 + 0^2 = 0^2 + (-1)^2$;

orange: $2 = 1^2 + 1^2 = (-1)^2 + 1^2 = (-1)^2 + (-1)^2 = 1^2 + (-1)^2$;

yellow: 3 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{3}$.

green: $4 = 2^2 + 0^2 = 0^2 + 2^2 = (-2)^2 + 0^2 = 0^2 + (-2)^2$;

blue:

$$5 = 2^2 + 1^2 = 1^2 + 2^2 = (-1)^2 + 2^2 = (-2)^2 + 1^2 = (-2)^2 + (-1)^2 = (-1)^2 + (-2)^2 = 1^2 + (- \\ = 2^2 + (-1)^2$$

;

purple: 6 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center (0,0) and radius √6.

violet: 7 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{7}$.

red: $8 = 2^2 + 2^2 = (-2)^2 + 2^2 = (-2)^2 + (-2)^2 = 2^2 + (-2)^2$;

orange: $9 = 3^2 + 0^2 = 0^2 + 3^2 = (-3)^2 + 0^2 = 0^2 + (-3)^2$;

yellow:

$$10 = 3^2 + 1^2 = 1^2 + 3^2 = (-1)^2 + 3^2 = (-3)^2 + 1^2 = (-3)^2 + (-1)^2 = (-1)^2 + (-3)^2 = 1^2 + (-3)^2 = 3^2 + (-1)^2;$$

green: 11 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(11)}$;

blue: 12 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(12)}$;

purple:

$$13 = 3^2 + 2^2 = 2^2 + 3^2 = (-2)^2 + 3^2 = (-3)^2 + 2^2 = (-3)^2 + (-2)^2 = (-2)^2 + (-3)^2 = 2^2 + (-3)^2 = 3^2 + (-2)^2;$$

violet: 14 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(14)}$.

red: 15 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center (0,0) and radius √(15);

orange: $16 = 4^2 + 0^2 = 0^2 + 4^2 = (-4)^2 + 0^2 = 0^2 + (-4)^2$;

yellow:

$$17 = 4^2 + 1^2 = 1^2 + 4^2 = (-1)^2 + 4^2 = (-4)^2 + 1^2 = (-4)^2 + (-1)^2 = (-1)^2 + (-4)^2 = 1^2 + (-4)^2 = 4^2 + (-1)^2;$$

green: $18 = 3^2 + 3^2 = (-3)^2 + 3^2 = (-3)^2 + (-3)^2 = 3^2 + (-3)^2$;

blue: 19 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(19)}$;

purple:

$$20 = 4^2 + 2^2 = 2^2 + 4^2 = (-2)^2 + 4^2 = (-4)^2 + 2^2 = (-4)^2 + (-2)^2 = (-2)^2 + (-4)^2 = 2^2 + (-4)^2 = 4^2 + (-2)^2;$$

violet: 21 cannot be written as $x^2 + y^2$, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(21)}$.

Graphic conclusions: For $n \in \{1,2,\ldots,21\}$, completely traversing the circles of equations $x^2 + y^2 = n$ (*curves with finite length*), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

$$1,2,4,5,8,9,10,13,16,17,18,20$$

can be written as $x^2 + y^2$, $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y) \in \mathbf{N} \times \mathbf{N}$ for the equation $x^2 + y^2 = n$, i.e. of those points on Ox_+, Oy_+ or in the I[st] quadrant that are on the circle and have coordinates in $\mathbf{N}$.(2)

Remark 1.3. Let $z = u + iv \in \mathbf{C}$. Then

$$u^2 + v^2 = |z|^2 = |z^2| = |u^2 - v^2 + i \cdot 2uv| = \sqrt{(u^2 - v^2)^2 + (2uv)^2}$$
$$\Leftrightarrow (u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2.$$

A method of generating, by a computer program, the numbers $n = x^2 + y^2 \in \mathbf{N}$ with form derived from Pythagorean numbers, can be obtained by giving values for $(u,v) \in \mathbf{N} \times \mathbf{N}$, with $u > v$ (for such a pair (u,v) we can construct accordingly and (v,u), and $(-u,v)$ and so on). Having obtained an infinite number of integers, the program must be stopped running, imposing an upper generation limit.

QUESTION 2. $a = 2$. What are integers $n \in \mathbf{Z}, n \geq 0$ that can be written as $x^2 + 2y^2$?

Answer 2.1. We partially researched, by **graphical analysis** and algebraic verification, which of the numbers $n \in \{0,1,2,\ldots,21\}$ can be written as $x^2 + 2y^2$.

1) n=0: We are looking if the ellipse that becomes a double point, with the equation

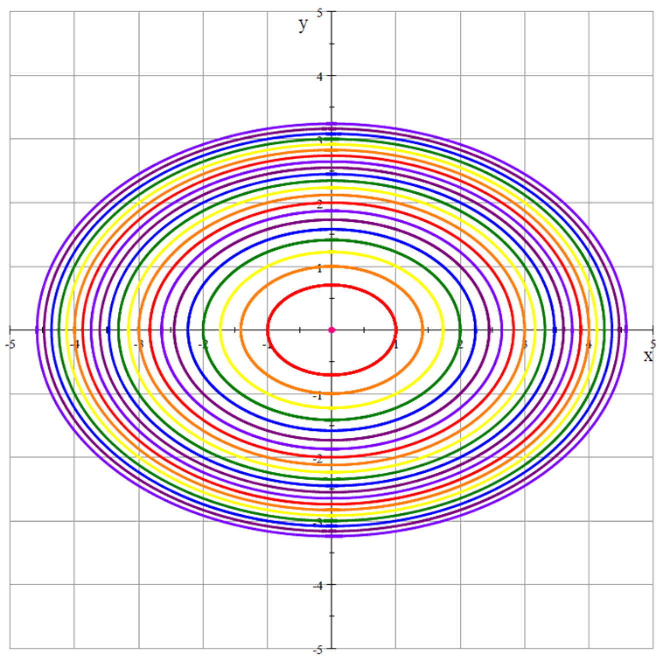
$$x^2 + 2y^2 = 0 \Leftrightarrow (x,y) = (0,0),$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. We find: $n = 0 = 0^2 + 2 \cdot 0^2$.(3)

2) $n \in \mathbf{Z}, n \geq 1$: We are looking if the ellipse with center $(0,0)$ and semiaxis $\sqrt{n}$, $\sqrt{(n/2)}$, with the equation

$$x^2 + 2y^2 = n \Leftrightarrow \left(\frac{x^2}{(\sqrt{n})^2}\right) + \left(\frac{y^2}{(\sqrt{n})^2}\right) = 1$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. By searching for $n \in \{1,2,\ldots,21\}$, i.e. looking for the colored ellipses passing through integer coordinate points, we find *all possible writing solutions*:



red: $1 = 1^2 + 2 \cdot 0^2 = (-1)^2 + 2 \cdot 0^2$;
orange: $2 = 0^2 + 2 \cdot 1^2 = 0^2 + 2 \cdot (-1)^2$;
yellow: $3 = 1^2 + 2 \cdot 1^2 = (-1)^2 + 2 \cdot 1^2 = (-1)^2 + 2 \cdot (-1)^2 = 1^2 + 2 \cdot (-1)^2$;
green: $4 = 2^2 + 2 \cdot 0^2 = (-2)^2 + 2 \cdot 0^2$;
blue: 5 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse with $x^2 + 2y^2 = 5$;
purple: $6 = 2^2 + 2 \cdot 1^2 = (-2)^2 + 2 \cdot 1^2 = (-2)^2 + 2 \cdot (-1)^2 = 2^2 + 2 \cdot (-1)^2$;
violet: 7 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2 + 2y^2 = 7$;

red: $8 = 0^2 + 2 \cdot 2^2 = 0^2 + 2 \cdot (-2)^2$;
orange: $9 = 3^2 + 2 \cdot 0^2 = 1^2 + 2 \cdot 2^2 = (-1)^2 + 2 \cdot 2^2 = (-3)^2 + 2 \cdot 0^2 = (-1)^2 + 2 \cdot (-2)^2$
$= 1^2 + 2(-2)^2$;
yellow: 10 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2 + 2y^2 = 10$;
green: $11 = 3^2 + 2 \cdot 1^2 = (-3)^2 + 2 \cdot 1^2 = (-3)^2 + 2 \cdot (-1)^2 = 3^2 + 2 \cdot (-1)^2$;
blue: $12 = 2^2 + 2 \cdot 2^2 = (-2)^2 + 2 \cdot 2^2 = (-2)^2 + 2 \cdot (-2)^2 = 2^2 + 2 \cdot (-2)^2$;
purple: 13 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2 + 2y^2 = 13$;
violet: 14 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points $(x,y$ located on the ellipse $x^2 + 2y^2 = 14$;
red: 15 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points $(x,y$ located on the ellipse $x^2 + 2y^2 = 15$;
orange: $16 = 4^2 + 2 \cdot 0^2 = (-4)^2 + 2 \cdot 0^2$;
yellow: $17 = 3^2 + 2 \cdot 1^2 = (-3)^2 + 2 \cdot 1^2 = (-3)^2 + 2 \cdot (-1)^2 = 3^2 + 2 \cdot (-1)^2$;
green: $18 = 4^2 + 2 \cdot 1^2 = 0^2 + 2 \cdot 3^2 = (-4)^2 + 2 \cdot 1^2 = (-4)^2 + 2 \cdot (-1)^2 =$
$0^2 + 2 \cdot (-3)^2 = 4^2 + 2 \cdot (-1)^2$;
blue: $19 = 1^2 + 2 \cdot 3^2 = (-1)^2 + 2 \cdot 3^2 = (-1)^2 + 2 \cdot (-3)^2 = 1^2 + 2 \cdot (-3)^2$;
purple: 20 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2 + 2y^2 = 20$
violet: 21 cannot be written as $x^2 + 2y^2$, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2 + 2y^2 = 21$.

Graphic conclusions: For $n \in \{1,2,\dots,21\}$, completely traversing ellipses of equations $x^2 + 2y^2 = n$ (*curves with finite length*), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

$$1,2,3,4,6,8,9,11,12,16,17,18,19$$

can be written as $x^2 + 2y^2$, $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y) \in \mathbf{N} \times \mathbf{N}$ for the equation $x^2 + 2y^2 = n$, i.e. of those points in the I[st] quadrant that are on the ellipse and have coordinates in $\mathbf{N}$.(4)

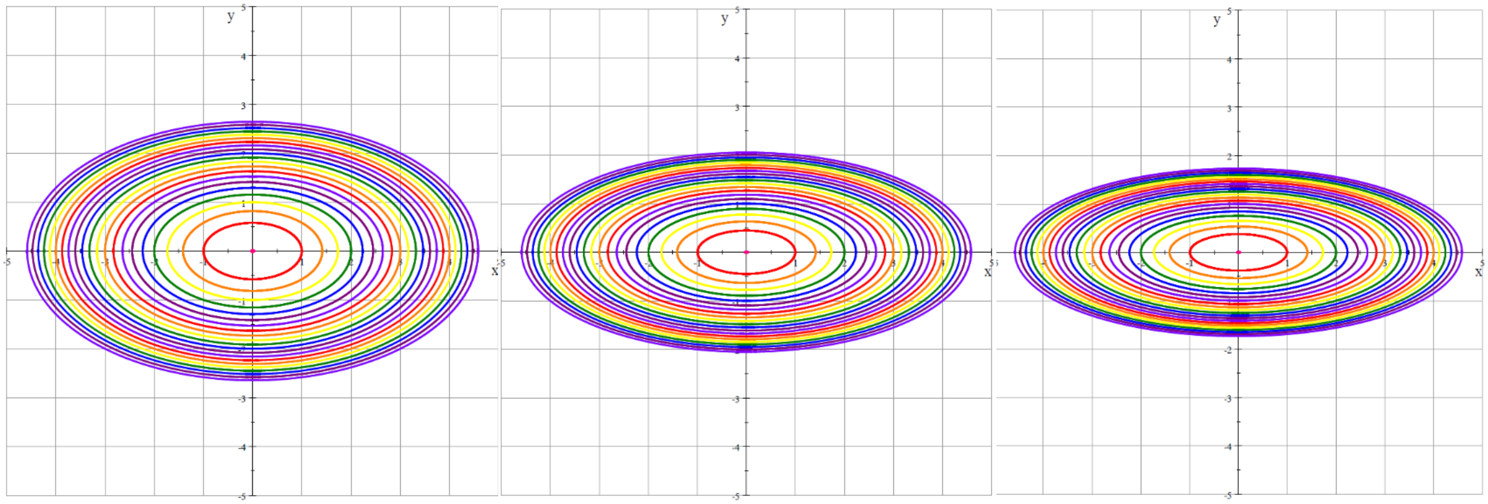
Remark 2.1. If $n = k^2, k \in \mathbb{Z}$ then there exists the trivial writing solution $n = k^2 + 2 \cdot 0^2 = (-k)^2 + 2 \cdot 0^2$.

QUESTION 3. $a = 3,5,7$. What are integers $n \in \mathbf{Z}, n \geq 0$ that can be written as $x^2 + ay^2$?

Answer 3.1. We partially researched, in the similar way, by **graphical analysis** and algebraic verification, which of the numbers $n \in \{0,1,2,\dots,21\}$ can be written as $x^2 + ay^2$.
Graphic conclusions: For $n = 0$, $0 = 0^2 + a \cdot 0^2$.(5) For $n \in \{1,2,\dots,21\}$, completely traversing ellipses of equations $x^2 + ay^2 = n$ (curves with finite length), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

$$a = 3: 1,3,4,7,9,12,13,16,19,21$$
$$a = 5: 1,4,5,6,9,14,16,20,21$$
$$a = 7: 1,4,7,8,9,11,16$$

can be written as $x^2 + ay^2, (x,y) \in \mathbf{Z} \times \mathbf{Z}$. In addition, the writing is not unique.(6)



Remark 3.1. If $n = k^2, k \in \mathbf{Z}$ then there exists the trivial writing solution $n = k^2 + a \cdot 0^2 = (-k)^2 + a \cdot 0^2$.

QUESTION 4. Certain $\boldsymbol{a > 0}$. What are integers $n \in \mathbf{Z}, n \geq 0$ that can be written as $\boldsymbol{x^2 + ay^2}$?

Answer 4.1. Graphical analysis and algebraic verification become difficult for large values of a>0 and n>0.(7)

Remark 4.1. If $n = k^2, k \in \mathbf{Z}$ then there exists the trivial writing solution $n = k^2 + a \cdot 0^2 = (-k)^2 + a \cdot 0^2$.

II. $\boldsymbol{a < 0}$ ($\boldsymbol{a = -1, -2}$, other cases)

Remark. We look for $n \in \mathbf{Z}$ for which there is $(x,y) \in \mathbf{Z} \times \mathbf{Z}$ such that $n = x^2 + ay^2$.

QUESTION 1. $\boldsymbol{a = -1}$. What are integers that can be written as $\boldsymbol{x^2 - y^2}$?

Answer 1.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n \in \{-21, \ldots, -2, -1, 0, 1, 2, \ldots, 21\}$ can be written as $x^2 - y^2$.

1) $n = 0$: We are looking if the hyperbola that becomes two secant lines, with the equation

$$x^2 - y^2 = 0 \Leftrightarrow (x = y \text{ or } x = -y) \quad (8)$$

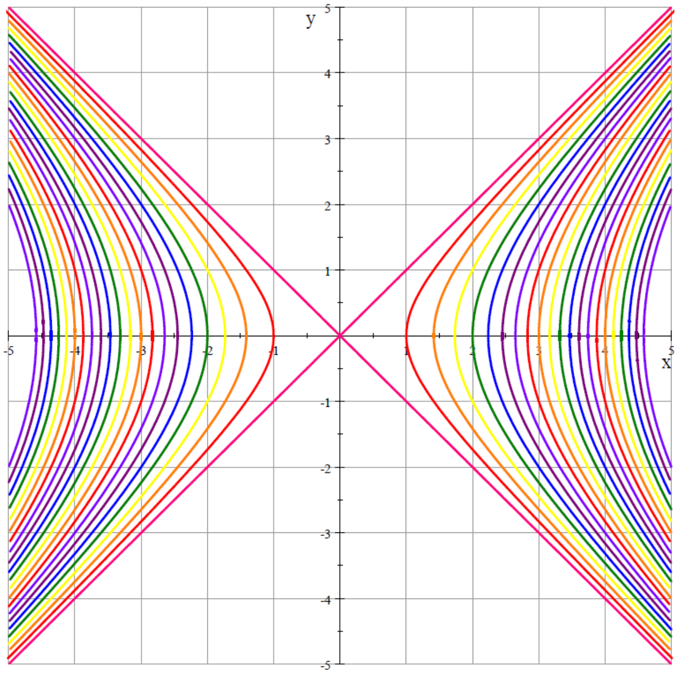
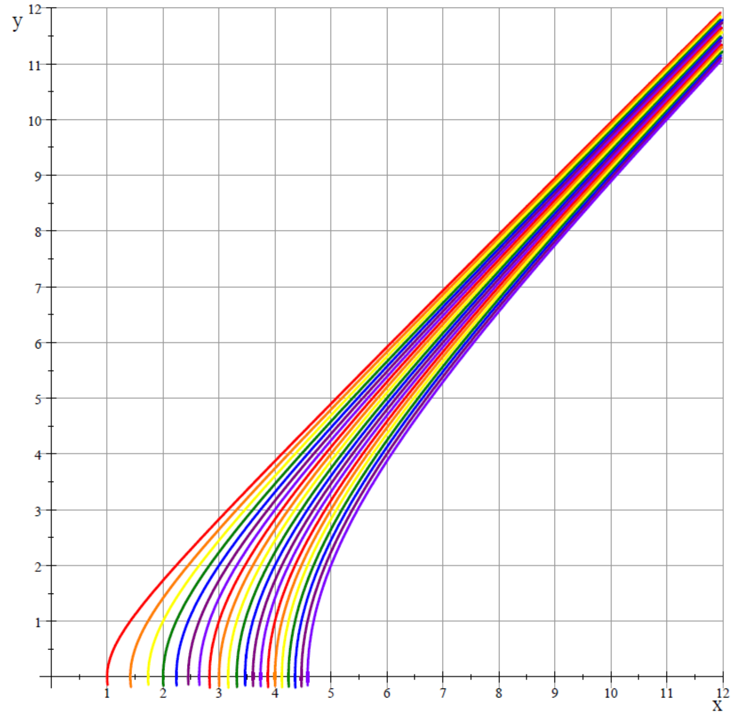
passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$.

We find: $n = 0 = 0^2 - 0^2 = \ldots = (\pm x)^2 - (\pm x)^2 = (\pm x)^2 - (\mp x)^2, \forall x \in \mathbf{Z}$.

2) $n \in \mathbf{Z}, n \geq 1$: We are looking if the hyperbola with the equation

$$x^2 - y^2 = n \Leftrightarrow \left(\frac{x^2}{(\sqrt{n})^2}\right) - \left(\frac{y^2}{(\sqrt{n})^2}\right) = 1$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5] \times [-5,5]$ only partially, relative to Ox_+, Oy_+ and I[st] quadrant, i.e. to $[0,12] \times [0,12]$, symmetrizing then the solutions found. By searching for $n \in \{1, 2, \ldots, 21\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only *some of the possible writing solutions,* those from the studied plan region:

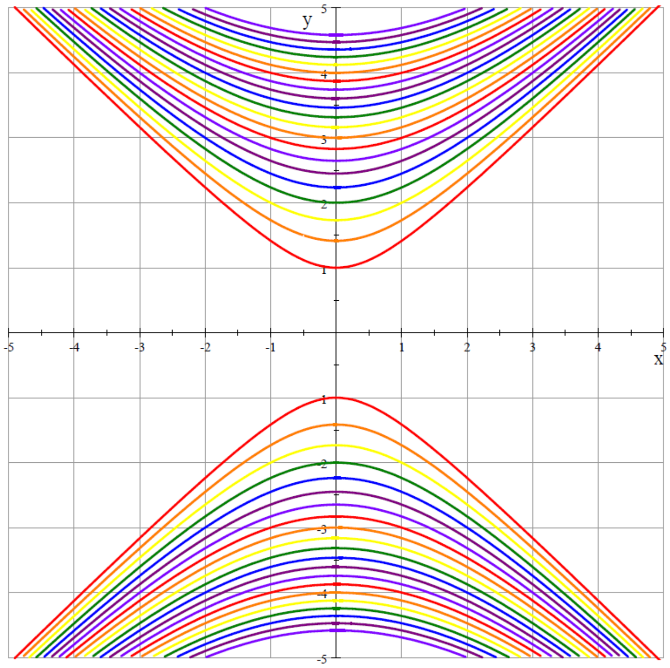
red: $1=1^2-0^2=(-1)^2-0^2$. (9)
orange: 2=we cannot decide from the study.
yellow: $3=2^2-1^2=(-2)^2-1^2=(-2)^2-(-1)^2=2^2-(-1)^2$.
green: $4=2^2-0^2=(-2)^2-0^2$.
blue: $5=3^2-2^2=(-3)^2-2^2=(-3)^2-(-2)^2=3^2-(-2)^2$.
purple: 6=we cannot decide from the study.
violet: $7=4^2-3^2=(-4)^2-3^2=(-4)^2-(-3)^2=4^2-(-3)^2$.
red: $8=3^2-1^2=(-3)^2-1^2=(-3)^2-(-1)^2=3^2-(-1)^2$.
orange: $9=5^2-4^2=(-5)^2-4^2=(-5)^2-(-4)^2=5^2-(-4)^2$.
yellow: 10=we cannot decide from the study.
green: $11=6^2-5^2=(-6)^2-5^2=(-6)^2-(-5)^2=6^2-(-5)^2$.
blue: $12=4^2-2^2=(-4)^2-2^2=(-4)^2-(-2)^2=4^2-(-2)^2$.
purple: $13=7^2-6^2=(-7)^2-6^2=(-7)^2-(-6)^2=7^2-(-6)^2$.
violet: 14=we cannot decide from the study.
red: $15=8^2-7^2=(-8)^2-7^2=(-8)^2-(-7)^2=8^2-(-7)^2$.
orange: $16=4^2-0^2=(-4)^2-0^2=5^2-3^2=(-5)^2-3^2=(-5)^2-(-3)^2=5^2-(-3)^2$.
yellow: $17=9^2-8^2=(-9)^2-8^2=(-9)^2-(-8)^2=9^2-(-8)^2$.
green: 18=we cannot decide from the study.
blue: $19=10^2-9^2=(-10)^2-9^2=(-10)^2-(-9)^2=10^2-(-9)^2$.
purple: $20=6^2-4^2=(-6)^2-4^2=(-6)^2-(-4)^2=6^2-(-4)^2$.
violet: $21=11^2-10^2=(-11)^2-10^2=(-11)^2-(-10)^2=11^2-(-10)^2$.

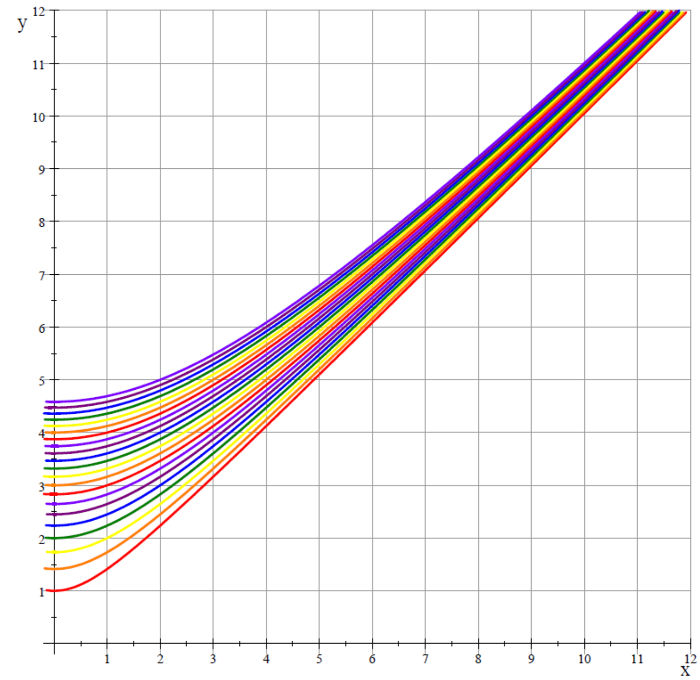
3) $n\in\mathbf{Z}, n\leq -1$: We are looking if the hyperbola with the equation

$$x^2-y^2=n \Leftrightarrow -\left(\frac{x^2}{(\sqrt{-n})^2}\right)+\left(\frac{y^2}{(\sqrt{-n})^2}\right)=1$$

passes through a point with integers coordinates $(x,y)\in\mathbf{Z}\times\mathbf{Z}$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+, Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{$

$-21,\ldots,-2,-1\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:





red: $-1 = 0^2 - 1^2 = 0^2 - (-1)^2$.
orange: 2=we cannot decide from the study.
yellow: $-3 = 1^2 - 2^2 = (-1)^2 - 2^2 = (-1)^2 - (-2)^2 = 1^2 - (-2)^2$.
green: $-4 = 0^2 - 2^2 = 0^2 - (-2)^2$.
blue: $-5 = 2^2 - 3^2 = (-2)^2 - 3^2 = (-2)^2 - (-3)^2 = 2^2 - (-3)^2$.
purple: -6=we cannot decide from the study.
violet: $-7 = 3^2 - 4^2 = (-3)^2 - 4^2 = (-3)^2 - (-4)^2 = 3^2 - (-4)^2$.
red: $-8 = 1^2 - 3^2 = (-1)^2 - 3^2 = (-1)^2 - (-3)^2 = 1^2 - (-3)^2$.
orange: $-9 = 4^2 - 5^2 = (-4)^2 - 5^2 = (-4)^2 - (-5)^2 = 4^2 - (-5)^2$.
yellow: -10=we cannot decide from the study.
green: $-11 = 5^2 - 6^2 = (-5)^2 - 6^2 = (-5)^2 - (-6)^2 = 5^2 - (-6)^2$.
blue: $-12 = 2^2 - 4^2 = (-2)^2 - 4^2 = (-2)^2 - (-4)^2 = 2^2 - (-4)^2$.
purple: $-13 = 6^2 - 7^2 = (-6)^2 - 7^2 = (-6)^2 - (-7)^2 = 6^2 - (-7)^2$.
violet: $-14 =$ we cannot decide from the study.
red: $-15 = 7^2 - 8^2 = (-7)^2 - 8^2 = (-7)^2 - (-8)^2 = 7^2 - (-8)^2$.
orange: $-16 = 0^2 - 4^2 = 0^2 - (-4)^2 = 3^2 - 5^2 = (-3)^2 - 5^2 = (-3)^2 - (-5)^2 = 3^2 - (-5)^2$.
yellow: $-17 = 8^2 - 9^2 = (-8)^2 - 9^2 = (-8)^2 - (-9)^2 = 8^2 - (-9)^2$.
green: $-18 =$ we cannot decide from the study.
blue: $-19 = 9^2 - 10^2 = (-9)^2 - 10^2 = (-9)^2 - (-10)^2 = 9^2 - (-10)^2$.
purple: $-20 = 4^2 - 6^2 = (-4)^2 - 6^2 = (-4)^2 - (-6)^2 = 4^2 - (-6)^2$.
violet: $-21 = 10^2 - 11^2 = (-10)^2 - 11^2 = (-10)^2 - (-11)^2 = 10^2 - (-11)^2$.

Graphic conclusions: 1) $n = 0 = (\pm x)^2 - (\pm x)^2 = (\pm x)^2 - (\mp x)^2, \forall x \in \mathbf{Z}$.(10)

2) For $n \in \{1,2,\ldots,21\}$, partially traversing hyperbolas of equations $x^2 - y^2 = n$ (*curves with infinite length*), from the point $(\sqrt{n},0)$ in the lower left to the upper right direction, we found that the numbers

$1,3,4,5,7,8,9,11,12,13,15,16,17,19,20,21$

can be written as $x^2-y^2,(x,y)\in\mathbf{Z}\times\mathbf{Z}$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in\mathbf{N}\times\mathbf{N}$ for the equation $x^2-y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in $\mathbf{N}$.

3) For $n\in\{-21,\ldots,-2,-1\}$, partially traversing hyperbolas of equations $x^2-y^2=n$ (curves with infinite length), from the point (0,√(-n)) in the lower left to the upper right direction, we found that the numbers

$$-1,-3,-4,-5,-7,-8,-9,-11,-12,-13,-15,-16,-17,-19,-20,-21$$

can be written as $x^2-y^2,(x,y)\in\mathbf{Z}\times\mathbf{Z}$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in\mathbf{N}\times\mathbf{N}$ for the equation $x^2-y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in $\mathbf{N}$.

Answer 1.2. Theorem in Number Theory.

a) If the integer $n\in\mathbf{Z}$ is of the form $2k+1,k\in\mathbf{Z},n\equiv 1(mod\ 2)$ or $4k,k\in\mathbf{Z},n\equiv 0(mod\ 4)$ then n can be written as $n=x^2-y^2,(x,y)\in\mathbf{Z}\times\mathbf{Z}$.

b) If $n\in\mathbf{Z}$ is of the form $4k+2,k\in\mathbf{Z},n\equiv 2(mod\ 4),$ the question remains open.

Sketch of proof. We look for $n\in\mathbf{Z}$ for which there exists $(x,y)\in\mathbf{Z}\times\mathbf{Z}$ such that

$$n=x^2-y^2\Leftrightarrow n=(x-y)(x+y).$$

$\forall n=2k+1,\ k\in\mathbf{Z}$ (so for $n=4k'+1,\ n=4k'+3,\ k'\in\mathbf{Z}$, $\exists x=k+1\in\mathbf{Z}$, $\exists y=k\in\mathbf{Z}$ such that $n=x^2-y^2$.

$\forall n=4k,\ k\in\mathbf{Z},\exists x=k+1\in\mathbf{Z},\ \exists y=k-1\in\mathbf{Z}$ such that $n=x^2-y^2$.

$\forall n=4k+2,\ k\in\mathbf{Z}$: the question remains open. We tried

$$x-y=2,\ x+y=2k+1\Rightarrow x=((2k+3)/2)\ \notin\mathbf{Z} \text{ and } y=((2k-1)/2)\ \notin\mathbf{Z}.$$

Remark 1.1. If $n=k^2,\ k\in\mathbf{Z}$ then there exists the trivial writing solution $n=k^2-0^2=(-k)^2-0^2$.
If $n=-k^2,\ k\in\mathbf{Z}$ then there exists the trivial writing solution $n=0^2-k^2=0^2-(-k)^2$.

QUESTION 2. $a=-2$. What are integers that can be written as x^2-2y^2?

Answer 2.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n\in\{-21,\ldots,-2,-1,0,1,2,\ldots,21\}$ can be written as x^2-2y^2.

1) $n=0$: We are looking if the hyperbola that becomes two secant lines, with the equation

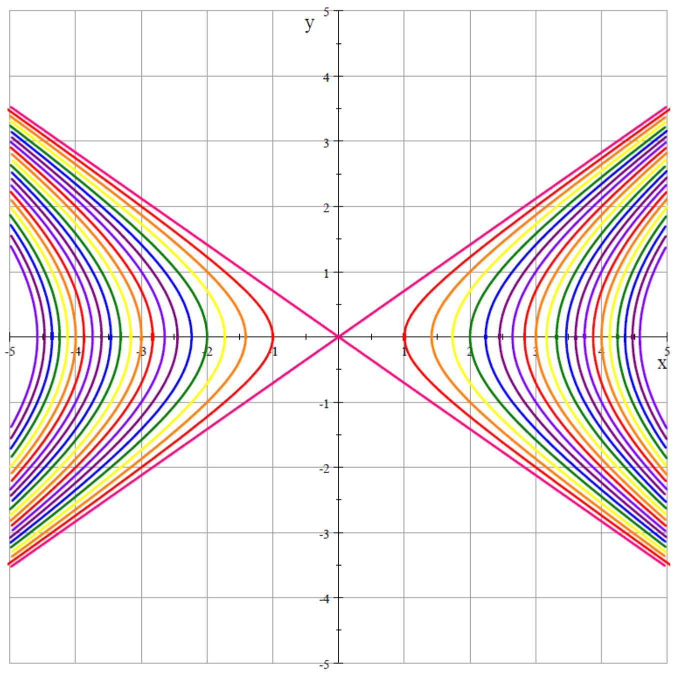
$$x^2-2y^2=0\Leftrightarrow(x=\sqrt{2}y\ or\ x=-\sqrt{2}y)$$

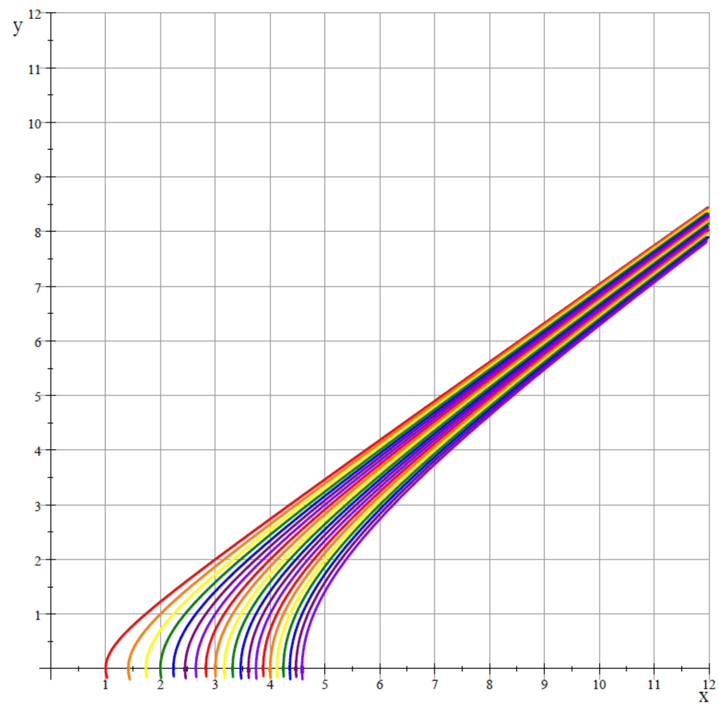
passes through a point with integers coordinates $(x,y)\in\mathbf{Z}\times\mathbf{Z}$. We find: $n=0=0^2-2\cdot 0^2$.(11)

2) $n\in Z,n\geq 1$: We are looking if the hyperbola with the equation

$$x^2-2y^2=n\Leftrightarrow\left(\frac{x^2}{(\sqrt{n})^2}\right)+\left(\frac{y^2}{(\sqrt{n/2})^2}\right)=1$$

passes through a point with integers coordinates $(x,y)\in\mathbf{Z}\times\boldsymbol{Z}$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+,Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{1,2,\ldots,21\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:





red: $1 = 1^2 - 2 \cdot 0^2 = (-1)^2 - 2 \cdot 0^2 = 3^2 - 2 \cdot 2^2 = (-3)^2 - 2 \cdot 2^2 = (-3)^2 - 2 \cdot (-2)^2 = 3^2 - 2 \cdot (-2)^2$.

orange: $2 = 2^2 - 2 \cdot 1^2 = (-2)^2 - 2 \cdot 1^2 = (-2)^2 - 2 \cdot (-1)^2 = 2^2 - 2 \cdot (-1)^2 = 10^2 - 2 \cdot 7^2 = (-10)^2 - 2 \cdot 7^2 = (-10)^2 - 2 \cdot (-7)^2 = 10^2 - 2 \cdot (-7)^2$.

yellow: 3=we cannot decide from the study.

green: $4 = 2^2 - 2 \cdot 0^2 = (-2)^2 - 2 \cdot 0^2 = 6^2 - 2 \cdot 4^2 = (-6)^2 - 2 \cdot 4^2 = (-6)^2 - 2 \cdot (-4)^2 = 6^2 - 2 \cdot (-4)^2$.

blue: 5=we cannot decide from the study.

purple: 6=we cannot decide from the study.

violet: $7 = 3^2 - 2 \cdot 1^2 = (-3)^2 - 2 \cdot 1^2 = (-3)^2 - 2 \cdot (-1)^2 = 3^2 - 2 \cdot (-1)^2 = 5^2 - 2 \cdot 3^2 = (-5)^2 - 2 \cdot 3^2 = (-5)^2 - 2 \cdot (-3)^2 = 5^2 - 2 \cdot (-3)^2$.

red: $8 = 4^2 - 2 \cdot 2^2 = (-4)^2 - 2 \cdot 2^2 = (-4)^2 - 2 \cdot (-2)^2 = 4^2 - 2 \cdot (-2)^2$.

orange: $9 = 3^2 - 0^2 = (-3)^2 - 0^2 = 9^2 - 2 \cdot 6^2 = (-9)^2 - 2 \cdot 6^2 = (-9)^2 - 2 \cdot (-6)^2 = 9^2 - 2 \cdot (-6)^2$.

yellow: 10=we cannot decide from the study.

green: 11=we cannot decide from the study.

blue: 12=we cannot decide from the study.

purple: 13=we cannot decide from the study.

violet: $14 = 4^2 - 2 \cdot 1^2 = (-4)^2 - 2 \cdot 1^2 = (-4)^2 - 2 \cdot (-1)^2 = 4^2 - 2 \cdot (-1)^2 = 8^2 - 2 \cdot 5^2 = (-8)^2 - 2 \cdot 5^2 = (-8)^2 - 2 \cdot (-5)^2 = 8^2 - 2 \cdot (-5)^2$.

red: 15=we cannot decide from the study.

orange: $16 = 4^2 - 2 \cdot 0^2 = (-4)^2 - 2 \cdot 0^2$.

yellow: $17 = 5^2 - 2 \cdot 2^2 = (-5)^2 - 2 \cdot 2^2 = (-5)^2 - 2 \cdot (-2)^2 = 5^2 - 2 \cdot (-2)^2 = 7^2 - 2 \cdot 4^2 = (-7)^2 - 2 \cdot 4^2 = (-7)^2 - 2 \cdot (-4)^2 = 7^2 - 2 \cdot (-4)^2$.

green: $18 = 6^2 - 2 \cdot 3^2 = (-6)^2 - 2 \cdot 3^2 = (-6)^2 - 2 \cdot (-3)^2 = 6^2 - 2 \cdot (-3)^2$.

blue: 19=we cannot decide from the study.

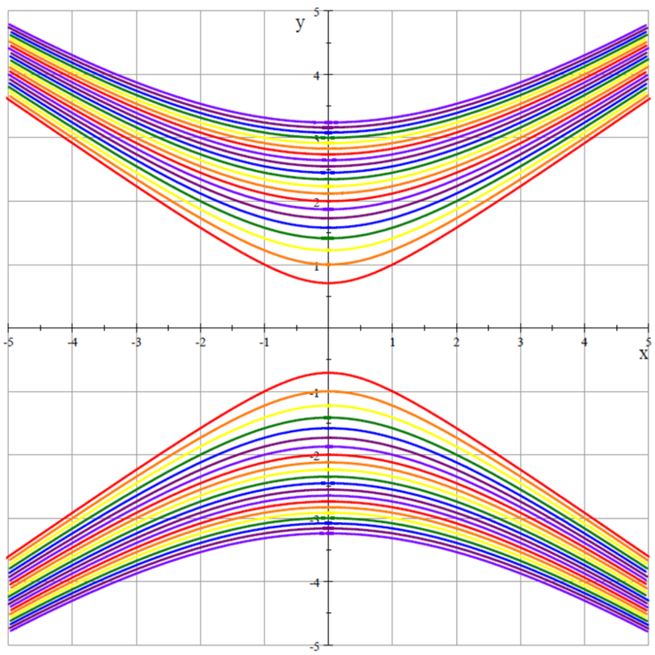
purple: 20=we cannot decide from the study.

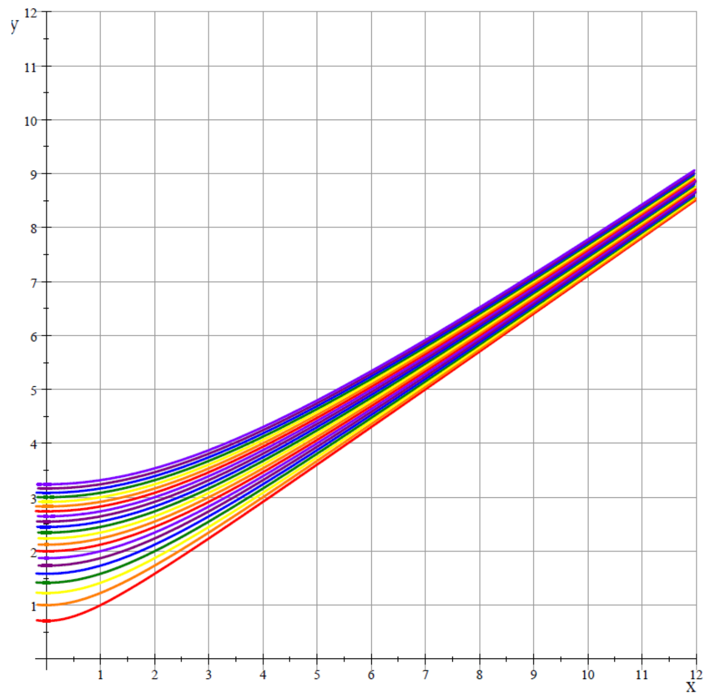
violet: 21=we cannot decide from the study.

3) $n \in \mathbf{Z}, n \leq -1$: We are looking if the hyperbola with center (0,0), with the equation

$$x^2 - 2y^2 = n \Leftrightarrow -\left(\frac{x^2}{(\sqrt{-n})^2}\right) + \left(\frac{y^2}{(\sqrt{-n/2})^2}\right) = 1$$

passes through a point with integers coordinates $(x,y) \in \mathbf{Z} \times \mathbf{Z}$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5] \times [-5,5]$ only partially, relative to Ox_+, Oy_+ and I[st] quadrant, i.e. to $[0,12] \times [0,12]$, symmetrizing then the solutions found. By searching for $n \in \{-21, \dots, -2, -1\}$ i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:





red: $-1 = 1^2 - 2 \cdot 1^2 = (-1)^2 - 2 \cdot 1^2 = (-1)^2 - 2 \cdot (-1)^2 = 1^2 - 2 \cdot (-1)^2 = 7^2 - 2 \cdot 5^2 = (-7)^2 - 2 \cdot 5^2 = (-7)^2 - 2 \cdot (-5)^2 = 7^2 - 2 \cdot (-5)^2$.

orange: -2 $= 0^2 - 2 \cdot 1^2 = 0^2 - 2 \cdot (-1)^2 = 4^2 - 2 \cdot 3^2 = (-4)^2 - 2 \cdot 3^2 = (-4)^2 - 2 \cdot (-3)^2 = 4^2 - 2 \cdot (-3)^2$.

yellow: -3=we cannot decide from the study.

green: $-4 = 2^2 - 2 \cdot 2^2 = (-2)^2 - 2 \cdot 2^2 = (-2)^2 - 2 \cdot (-2)^2 = 2^2 - 2 \cdot (-2)^2$.

blue: -5=we cannot decide from the study.

purple: -6=we cannot decide from the study.

violet: $-7 = 5^2 - 2 \cdot 4^2 = (-5)^2 - 2 \cdot 4^2 = (-5)^2 - 2 \cdot (-4)^2 = 5^2 - 2 \cdot (-4)^2 = 11^2 - 2 \cdot 8^2 = (-11)^2 - 2 \cdot 8^2 = (-11)^2 - 2 \cdot (-8)^2 = 11^2 - 2 \cdot (-8)^2$.

red: $-8 = 0^2 - 2 \cdot 2^2 = 0^2 - 2 \cdot (-2)^2 = 8^2 - 2 \cdot 6^2 = (-8)^2 - 2 \cdot 6^2 = (-8)^2 - 2 \cdot (-6)^2 = 8^2 - 2 \cdot (-6)^2$.

orange: $-9 = 3^2 - 2 \cdot 3^2 = (-3)^2 - 2 \cdot 3^2 = (-3)^2 - 2 \cdot (-3)^2 = 3^2 - 2 \cdot (-3)^2$.

yellow: -10=we cannot decide from the study.

green: -11=we cannot decide from the study.

blue: -12=we cannot decide from the study.

purple: -13=we cannot decide from the study.

violet: $-14 = 2^2 - 2 \cdot 3^2 = (-2)^2 - 2 \cdot 3^2 = (-2)^2 - 2 \cdot (-3)^2 = 2^2 - 2 \cdot (-3)^2 = 6^2 - 2 \cdot 5^2 = (-6)^2 - 2 \cdot 5^2 = (-6)^2 - 2 \cdot (-5)^2 = 6^2 - 2 \cdot (-5)^2$.

red: -15=we cannot decide from the study.
orange: $-16=0^2-2\cdot 4^2=0^2-2\cdot(-4)^2=4^2-2\cdot 4^2=(-4)^2-2\cdot 4^2=(-4)^2-2\cdot(-4)^2$ $=4^2-2\cdot(-4)^2$.

yellow: $-17=1^2-2\cdot 3^2=(-1)^2-2\cdot 3^2=(-1)^2-2\cdot(-3)^2=1^2-2\cdot(-3)^2=9^2-2\cdot 7^2$ $=(-9)^2-2\cdot 7^2=(-9)^2-2\cdot(-7)^2=9^2-2\cdot(-7)^2$.
green: $-18=0^2-2\cdot 3^2=0^2-2\cdot(-3)^2$.
blue: -19=we cannot decide from the study.
purple: -20=we cannot decide from the study.
violet: -21=we cannot decide from the study.

Graphic conclusions:
1) $n=0=0^2-2\cdot 0^2$.(12)
2) For $n\in\{1,2,...,21\}$, partially traversing hyperbolas of equations $x^2-2y^2=n$ (*curves with infinite length*), from the point $(\sqrt{n},0)$ in the lower left to the upper right direction, we found that the numbers

$$1,2,4,7,8,9,14,16,17,18$$

can be written as $x^2-2y^2,(x,y)\in\mathbf{Z}\times\mathbf{Z}$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in\mathbf{N}\times\mathbf{N}$ for the equation $x^2-2y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in $\mathbf{N}$.
3) For $n\in\{-21,...,-2,-1\}$, partially traversing hyperbolas of equations $x^2-2y^2=n$ (curves with infinite length), from the point $(0,\sqrt{(-n)})$ in the lower left to the upper right direction, we found that the numbers

$$-1,-2,-4,-7,-8,-9,-14,-16,-17,-18$$

can be written as as $x^2-2y^2,(x,y)\in\mathbf{Z}\times\mathbf{Z}$.. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in\mathbf{N}\times\mathbf{N}$ for the equation $x^2-2y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in $\mathbf{N}$.

QUESTION 3. Certain $a<0$. What are integers that can be written as x^2+ay^2?

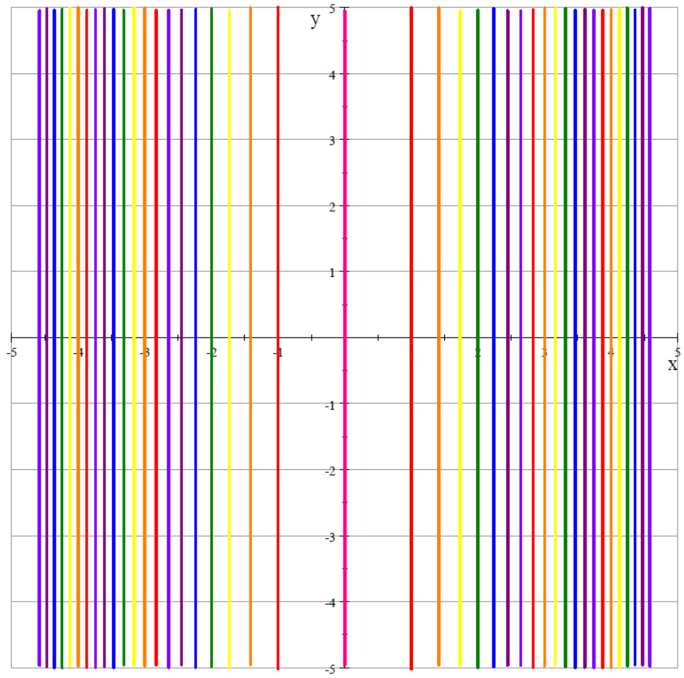
Answer 3.1. Graphical analysis and algebraic verification become difficult for large values of |a|>0 and |n|>0. For the the prime number $n=p>0$, or $n=p<0$ the reader can look in the bibliography below (Ionaşcu, Patterson [7])

III. a=0

QUESTION 1. $a=0$. What are integers that can be written as $x^2+0\cdot y^2$?

Remark. Solution. Since $x^2+0\cdot y^2\geq 0,\forall(x,y)\in\mathbf{Z}\times\mathbf{Z}\Rightarrow n\in\mathbf{Z},n<0$ cannot be written as $x^2+0\cdot y^2$, $(x,y)\in\mathbf{Z}\times\mathbf{Z}$.
We look for $n\in\mathbf{Z},n\geq 0$ for which there is $(x,y)\in\mathbf{Z}\times\mathbf{Z}$ such that $n=x^2+0\cdot y^2$.
Answer 1.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n\in\{0,1,2,...,21\}$ can be written as $x^2+0\cdot y^2$.
1) $n=0$: We are looking if the two coincident lines, with the equation $x^2=0\Leftrightarrow x=0$ passes through a point with integers coordinates $(x,y)\in\mathbf{Z}\times\mathbf{Z}$. We find: $n=0=0^2+0\cdot 0^2=0^2+0\cdot y^2,\forall y\in\mathbf{Z}$.
2) $n\in\mathbf{Z},n\geq 1$: We are looking if the lines with the equation $x^2+0\cdot y^2=n\Leftrightarrow(x=\sqrt{n}\ or\ x=-\sqrt{n})$ passes through a point with integers coordinates $(x,y)\in\mathbf{Z}\times\mathbf{Z}$. By searching for $n\in\{1,2,...,21\}$, i.e. looking for the colored lines passing through integer coordinate points, we observe only *some of the possible writing solutions*, those from the studied plan region:
red: $1=1^2+0\cdot y^2=(-1)^2+0\cdot y^2,y\in\mathbf{Z}$

orange: 2 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{2}$ or $x=-\sqrt{2}$.
yellow: 3 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{3}$ or $x=-\sqrt{3}$.
green: $4=2^2+0\cdot y^2=(-2)^2+0\cdot y^2, y\in \mathbf{Z}$
blue: 5 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{5}$ or $x=-\sqrt{5}$.



purple: 6 it cannot be written as $x^2+0\cdot y^2$...
violet: 7 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{7}$ or $x=-\sqrt{7}$.
red: 8 it cannot be written as $x^2+0\cdot y^2$...
orange: $9=3^2+0\cdot y^2=(-3)^2+0\cdot y^2, y\in \mathbf{Z}$
yellow: 10 it cannot be written as $x^2+0\cdot y^2$...
green: 11 it cannot be written as $x^2+0\cdot y^2$...
blue: 12 it cannot be written as $x^2+0\cdot y^2$...
purple: 13 it cannot be written as $x^2+0\cdot y^2$...
violet: 14 it cannot be written as $x^2+0\cdot y^2$...
red: 15 it cannot be written as $x^2+0\cdot y^2$...
orange: $16=4^2+0\cdot y^2=(-4)^2+0\cdot y^2, y\in Z$
yellow: 17 it cannot be written as $x^2+0\cdot y^2$...
green: 18 it cannot be written as $x^2+0\cdot y^2$...
blue: 19 it cannot be written as $x^2+0\cdot y^2$...
purple: 20 it cannot be written as $x^2+0\cdot y^2$...
violet: 21 it cannot be written as $x^2+0\cdot y^2$...

Graphic conclusions: For $n\in\{1,2,...,21\}$, partially traversing hlines of equations $x^2+0\cdot y^2=n$ (*curves with infinite length*), from bottom to top direction, we found that the numbers

$$1,4,9,16$$

can be written as $x^2+0\cdot y^2, (x,y)\in \mathbf{Z}\times\mathbf{Z}$. In addition, the writing is not unique (13)

Bibliography **:** The reader who wishes to know more about this problem in number theory, she and he can find many results in the following papers.

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Notes d'édition
(1) This case can be solved directly , without a graphical analysis: the sum of two non negative numbers is equal to 0 if and only onf the 2 numbers are equal to 0.
(2) All these results are easily found without a graphical analysis: if $x^2+y^2=n$ then $x^2 \leq n$ and $y^2 \leq n$. There are only a finite number of couple $(x,y) \in \mathbf{Z}$ such that then $x^2 \leq n$ and $y^2 \leq n$. With n≤21 it is very to check if there exists n such that $x^2+y^2=n$. For instance, if $n=15$ then $x \in \{1,2,3\}$ and $y \in \{1,2,3\}$ and there is no solution.
(3) See note (1)
(4) All these results are easily found without a graphical analysis: if $x^2+2y^2=n$ then $x^2 \leq n$ and $2y^2 \leq n$. There are only a finite number of couple $(x,y) \in \mathbf{Z}$ such that then $x^2 \leq n$ and $2y^2 \leq n$. With $n \leq 21$ it is very to check if there exists n such that $x^2+2y^2=n$. For instance, if $n=21$ then $x \in \{1,2,3,4\}$ and $y \in \{1,2,3\}$ and there is no solution.
(5) See note (1)
(6) All these results are easily found without a graphical analysis: if $x^2+ay^2=n$ with a>0 then $x^2 \leq n$ and $ay^2 \leq n$. There are only a finite number of couple $(x,y) \in \mathbf{Z}$ such that then $x^2 \leq n$ and $ay^2 \leq n$. With $n \leq 21$ it is very to check if there exists n such that $x^2+y^2=n$. For instance, if $n=15$ and $a=3$ then $x \in \{1,2,3\}$ and $y \in \{1,2\}$ and there is no solution.
(7) For the prime numbers $n=p>0$, some results are known : see the bibliography above.
(8) The graphical analysis is not necessary in this case.
(9) The graphical analysis is not necessary in this case.
(10) The graphical analysis is not necessary in this case.
(11) The graphical analysis is not necessary in this case.
(12) The graphical analysis is not necessary in this case.
(13) We can go to this conclusion much quicky without a graphical analysis: the integers n which can be written as $n= y^2$ are square of an integer. If $n \in \{1,2,\dots,21\}$, $n=1,4,9,16$, as $1,4,9,16$ are the only squares in $\{1,2,\dots,21\}$