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A constant sum

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The subject:

A convex polygon is called balanced if it has the following property : the sum of the distances from one interior point to its sides (or extensions of its sides) does not depend on the position of the point (it is constant). Can you find which polygons are balanced ?

Introduction:

The main target was to find a generalization, or somewhat of a generalization, as it was impossible to study each polygon. What we tried to do was to first check if some particular polygons were balanced, observe what properties they shared, and find a polygon which also shared them, thus leading to a generalization.

Results: We wanted to find as many good cases as possible and be able to fit them into a single category, therefore, throughout our proofs, we showed that regular polygons and parallelograms are 'balanced'.

Content

1. Regular polygons:
   a) equilateral triangle;
   b) square;
   c) rhombus/diamond;
   d) regular hexagon;
   e) generalization with n sides.
2. Rectangle;
3. A triangle is balanced if and only if it is equilateral (1)
4. The parallelogram is balanced;
5. A quadrilateral is balanced if and only if it is a parallelogram.

1. Regular polygons

a) Equilateral triangle

- ***Hypothesis***: We know that ΔABC is equilateral;
- ***Conclusion***: $h_1+h_2+h_3$ is constant.

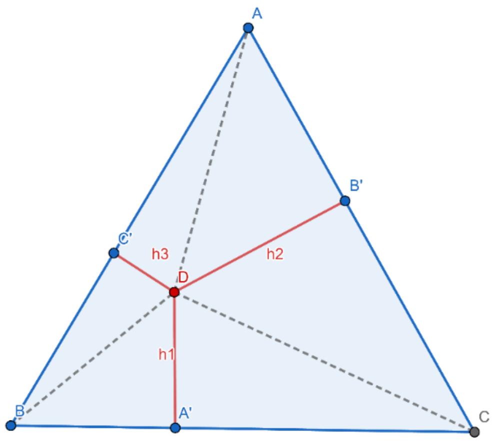

fig 1

- ***Demonstration***:

We consider $AB = BC = AC = l$ (the side of the triangle);

$A_{ABC}= A_{OAB}+ A_{OBC}+ A_{OAC}=\frac{h_1 \cdot l}{2}+\frac{h_2 \cdot l}{2}+\frac{h_3 \cdot l}{2}$, therefore

$h_1+h_2+h_3=\frac{2 \cdot A_{ABC}}{l}$ is constant.

b) Square

- ***Hypothesis:***We know that ABCD is a square;
- ***Conclusion***: $h_1 + h_2 + h_3 + h_4$ is constant;

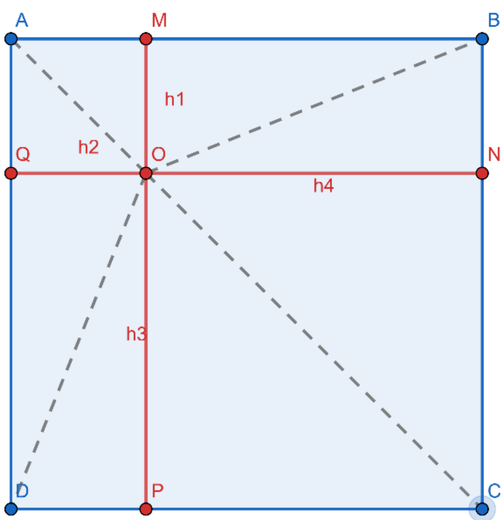

fig 2

- ***Demonstration:***

$AB = BC = CD = DA = l$ (the side of the square);

$h_1 + h_3 = h_2 + h_4 = l$;

$A_{ABCD}=A_{BOA} + A_{DOA} + A_{DOC} + A_{BOC}$;

$A_{ABCD}=\frac{h_1 \cdot l}{2} + \frac{h_2 \cdot l}{2} + \frac{h_3 \cdot l}{2} + \frac{h_4 \cdot l}{2}= \frac{l(h_1+h_2+h_3+h_4)}{2}$,

so $h_1 + h_2 + h_3 + h_4=\frac{2 \cdot A_{ABCD}}{l}$ is constant.

c) Rhombus/Diamond

- ***Hypothesis:*** We know that ABCD is a rhombus;
- ***Conclusion***: $h_1 + h_2 + h_3 + h_4$ is constant;

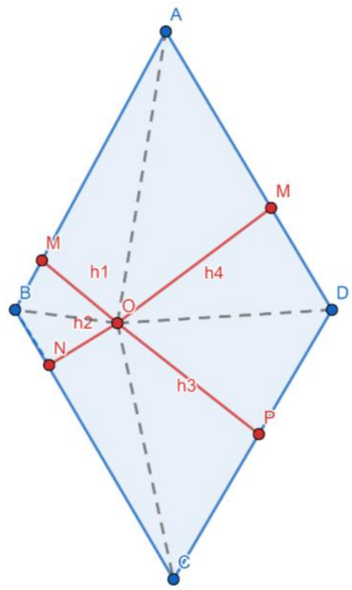

fig 3

- ***Demonstration:***

$AB = BC = CD = DA = l$ (the side of the rhombus);

$A_{ABCD}=A_{BOA} + A_{DOA} + A_{DOC} + A_{BOC}$;

$A_{ABCD} = \frac{h_1 \cdot l}{2} + \frac{h_2 \cdot l}{2} + \frac{h_3 \cdot l}{2} + \frac{h_4 \cdot l}{2} = \frac{l(h_1+h_2+h_3+h_4)}{2}$;

therefore $h_1 + h_2 + h_3 + h_4 = \frac{2 \cdot A_{ABCD}}{l}$ is constant.

d) Regular hexagon

- ***Hypothesis***: We know that ABCDEF is a regular hexagon;
- ***Conclusion***: $h_1 + h_2 + h_3 + h_4 + h_5 + h_6$ is constant;

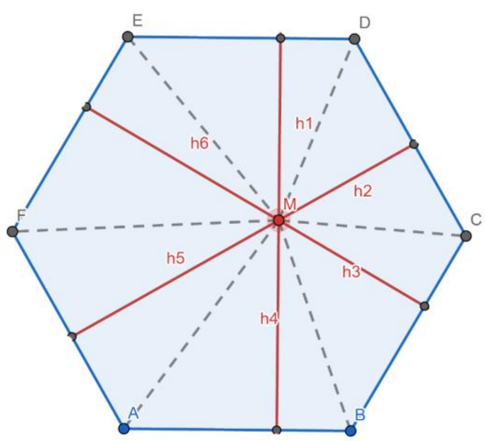

fig 4

- ***Demonstration:***

$AB = BC = CD = DE = EF = FA = l$ (the side of the hexagon)

$A_{ABCDEF}=A_{AMB}+A_{CMB}+A_{CMD}+A_{EDM}+A_{EFM}+A_{FMA}$

$A_{ABCDEF}=\frac{h_1 \cdot l}{2} + \frac{h_2 \cdot l}{2} + \frac{h_3 \cdot l}{2} + \frac{h_4 \cdot l}{2} + \frac{h_5 \cdot l}{2} + \frac{h_6 \cdot l}{2}$

$= \frac{l(h_1+h_2+h_3+h_4+h_5+h_6)}{2}$,

hence $h_1 + h_2 + h_3 + h_4 + h_5 + h_6 = \frac{2 \cdot A_{ABCDEF}}{l}$ is constant

e) Generalization

- ***Hypothesis***: We know that $A_1A_2\ldots A_n$ is a regular polygon.
- ***Conclusion***: $h_1 + h_2 + h_3 \ldots \ldots + h_n$ is constant;

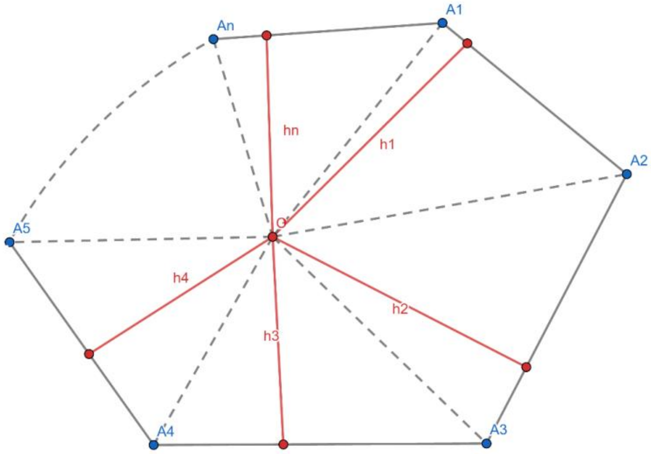

fig 5

- ***Demonstration***:

$A_1A_2=A_2A_3=\ldots=A_nA_1 = l$ (the side of the polygon);

$A_{A_1A_2\ldots A_n}=A_{A_1OA_2}+A_{A_2OA_3}+\ldots+A_{A_{n-1}OA_n}+A_{A_nOA_1}$;

$$A_{A_1A_2\ldots A_n}=\frac{h_1\cdot l}{2}+\frac{h_2\cdot l}{2}+\frac{h_3\cdot l}{2}+\ldots+\frac{h_n\cdot l}{2}=\frac{l(h_1+h_2+h_3+\cdots+h_n)}{2}$$

Thus $h_1+h_2+h_3+\cdots+h_n=\frac{2\cdot A_{A_1A_2\ldots A_n}}{l}$ is constant (2)

2. Rectangle

- ***Hypothesis***: We know that ABCD is a rectangle;
- ***Conclusion***: Rectangles are balanced.

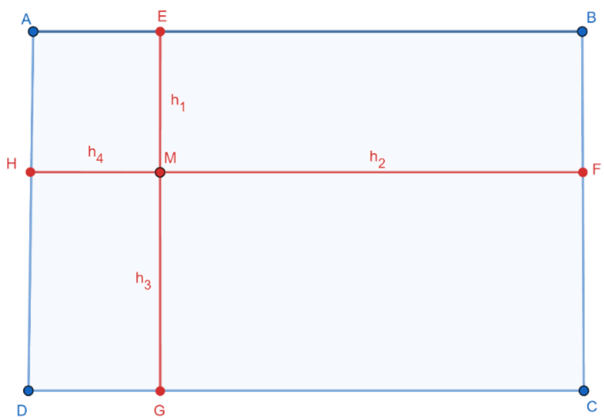

fig 6

- ***Demonstration***

We know that ME⊥AB, MG⊥DC, MF⊥BC and MH⊥AD.

ME⊥AB and FB⊥AB, hence ME∥FB, but we also know that F belongs to BC, therefore ME∥BC.

MG⊥DC and FC⊥DC, hence MG∥FC, but we also know that F belongs to BC, therefore MG∥BC.

ME∥BC and MG∥BC, so M, E and G are collinear (Euclid's axiom) and EG=BC= l (the width of the rectangle).

MH⊥AD and EA⊥AD, hence MH∥AE, but we also know that E belongs to AB, therefore MH∥AB.

MF⊥BC and BE⊥BC, hence MF∥EB, but we also know that E belongs to AB, therefore MF∥AB.

MH∥AB and MF∥AB, so therefore H, M and F are collinear (Euclid's axiom) and HF=CD=L (the length of the rectangle)
$h_1 + h_2+h_3 + h_4=L + l$ is constant, hence ABCD is balanced.

3. If a triangle is balanced, then the triangle is equilateral

- ***Hypothesis***: $MM_1+MM_2+MM_3$ is constant, for any M point from the interior du triangle ABC (MM_1, MM_2, MM_3 being the distances from M to the triangle's sides).
- ***Conclusion***: ΔABC is equilateral;

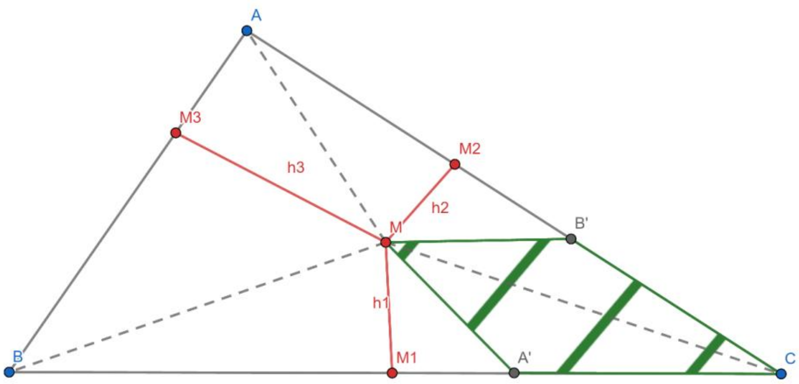

fig7

- ***Demonstration***:

If AB<BC and AB<AC, we choose **A'** and **B'** on the sides of the triangle so that **AB'=AB=BA'**;

We know that $MM_1+MM_2+MM_3$ is constant, so $\frac{AB}{2}\cdot(MM_1+MM_2+MM_3)$ is constant. Therefore $\frac{AB}{2}\cdot(h_1+h_2+h_3)$ is constant.

$A_{ABA'MB'}=A_{AMB}+A_{AMB'}+A_{A'MB}=\frac{h_1\cdot A'B}{2}+\frac{h_2\cdot AB'}{2}+\frac{h_3\cdot AB}{2}$ and AB=A'B=AB', therefore

$A_{ABA'MB'}=\frac{AB}{2}\cdot(h_1+h_2+h_3)$ = constant, so $A_{MA'CB'}=A_{ABC}-A_{ABA'MB}$ is constant;

But C, A′, B′are fixed points, whereas M is not, hence $A_{MA'CB'}$ can not be constant, thus we know the conjecture is not true for any random triangle.

Next, we wanted to prove that it is not enough for a triangle to be isosceles; therefore it needs to be equilateral.

- ***Isosceles triangle***

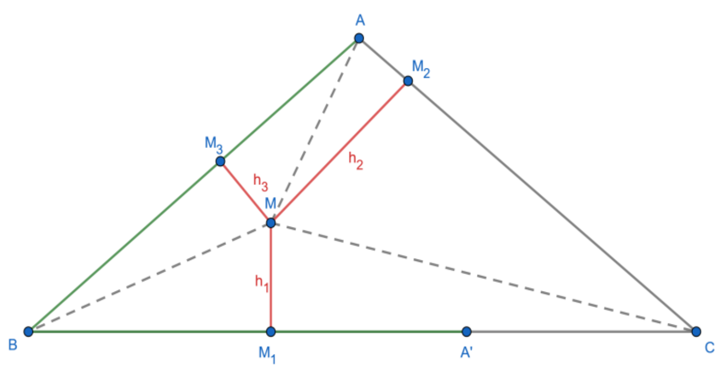

fig 8

If ΔABC is isosceles (AB=AC) (3), then B'=C and we know that $\frac{AB}{2}\cdot(h_1+h_2+h_3)$ is constant, therefore $A_{ABA'MC}$ is also constant; From here, we can also deduce the following:

- $A_{MA'C}$ is constant so $h_1\cdot$ A'C is constant, so h_1 is constant;
- The triangle is balanced whenever M is on a parallel to BC (4)

- The hypothesis is not valid for every point inside the triangle.
  Thus, the triangle can't be isosceles.

We have now proved that the conjecture isn't true for any triangle and the triangle can't be isosceles, so ΔABC **must be equilateral.** (C=A′=B′, so $A_{MA'CB'}=0$ and it is constant)

4. Parallelograms are balanced

- ***Hypothesis***: We know that ABCD is a parallelogram;
- ***Conclusion***: Parallelograms are balanced.

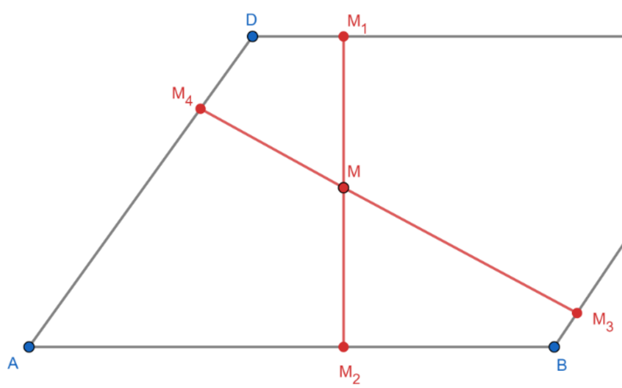

fig 9

- ***Demonstration***:

AB∥ CD, so $MM_1 + MM_2$ is constant (the distance between two parallel lines is constant).

AD ∥ BC, therefore $MM_3 + MM_4$ is constant as well.

According to the previous statements, $MM_1 + MM_2 + MM_3 + MM_4$ is constant, for any M from the interior of the parallelogram;

5. A quadrilateral is balanced, then it is a parallelogram.

- ***Hypothesis***: ABCD is a convex quadrilateral and $h_1 + h_2 + h_3 + h_4$ is constant, for any M from the interior of the triangle;
- ***Conclusion:*** ABCD can only be a parallelogram;

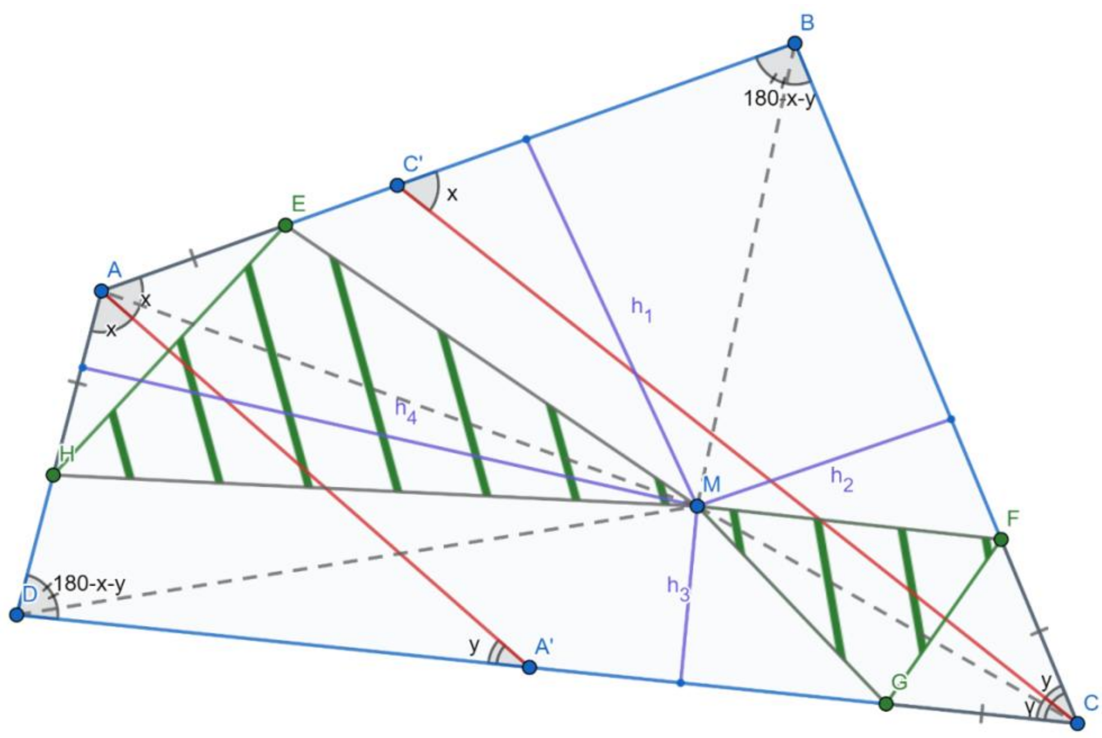

fig 10

- ***Demonstration:***

Let E, F, G, H be points on the sides of the triangle so that AH=AE=CF=CG;

A_{AEMH}=A_{AEM} +A_{AMH} A_{AEMH}= $\frac{h_1 \cdot \mathrm{AE}}{2}$ + $\frac{h_4 \cdot AH}{2}$and AE= AH, therefore A_{AEMH}=$\frac{AE\cdot (h_1+h_4)}{2}$;
A_{CGMF}= A_{CGM}+A_{CMF}, A_{CGMF}= $\frac{h_2 \cdot \mathrm{CF}}{2}$ + $\frac{h_3 \cdot CG}{2}$ and CF = CG = AE, therefore A_{CGMF} = $\frac{AE(h_2+h_3)}{2}$;

Thus , A_{AEMH}+A_{CGMF}= $\frac{AE\cdot (h_1+h_2+h_3+h_4)}{2}$ is constant;

We also know that A_{CGF}+A_{AEH}=$\frac{CF\cdot CG\cdot sin(\measuredangle FCG)}{2}$+$\frac{AH\cdot AE\cdot \sin(\measuredangle EAH)}{2}$. But CF=CG=AH=AE= a (notation),
therefore A_{CGF}+A_{AEH} $= \frac{a^2}{2} \cdot (sin(\measuredangle C) + \sin(\measuredangle A))$, which is a constant.

Hence , A_{EMH}+A_{GMF} $= (A_{AEMH} - A_{AEH}) + (A_{CGMF} - A_{CGF})$ is constant , for any M from the interior of EFGH (**1**);

If EH∦FG, we will consider the quadrilateral EHGF, EH∩ FG= {S};

Let X, Y be points (X ∈ FG and $Y \in HE$) so that SX=GF and SY= HE (**2**);

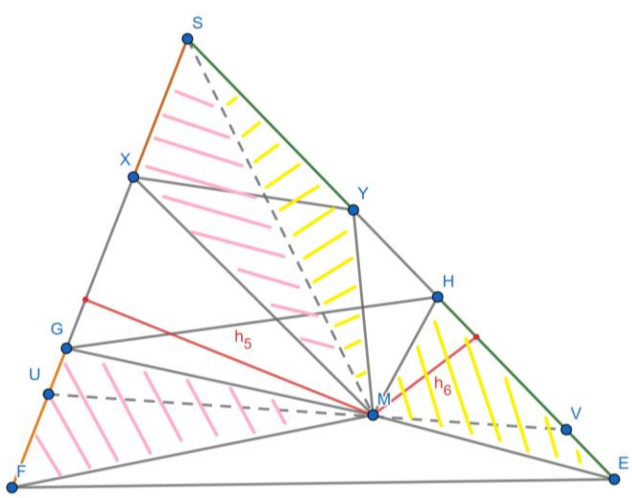

fig 11

In this drawing, h_5 is the distance from M to GF, and h_6 is the distance from M to EH.

A_{GMF}= $\frac{h_5 \cdot \mathrm{GF}}{2}$, A_{SXM}= $\frac{h_5 \cdot \mathrm{SX}}{2}$and GF=SX (from point 2), therefore A_{GMF}=A_{SXM};

A_{MHE}= $\frac{h_6 \cdot \mathrm{HE}}{2}$, A_{SMY}= $\frac{h_6 \cdot \mathrm{SY}}{2}$ and HE=SY, therefore A_{MHE}=A_{SMY};

A_{MHE}+ A_{GMF} is constant (from point 1),
so A_{SXM}+ A_{SMY} is constant;

A_{SXM}+ A_{SMY}= A_{SXY}+A_{MXY} andA_{SXY} is constant (S, X, Y fixed points), consequently A_{MXY} is constant;

A_{MXY}=$\frac{\mathrm{d}\,(\mathrm{M,XY}) \cdot \mathrm{XY}}{2}$ and XY is constant, so d (M, XY) = constant. Therefore M∈ (UV), where UV ∥ XY, U∈ GF, V∈ HE, which means that M is not a random point that belongs to the interior of the quadrilateral if EH ∦ FG;

In conclusion, EH ∥ FG;

We choose (AA′ bisector of $\measuredangle DAB$. We know that ΔAHE isosceles, so (AA′ is the perpendicular bisector [EH]; (in an isosceles triangle, the perpendicular bisector, the bisector, the height and the median on the base, from the vertex opposed to it all correspond)

We choose (CC′ bisector of $\measuredangle DCB$. We know that ΔCGF isosceles, so (CC′- perpendicular bisector [GF]; (in an isosceles triangle, the perpendicular bisector, the bisector, the height and the median on the base, from the vertex opposed to it all correspond)

AA′ ⊥ EH, CC′⊥ GF and EH ∥ GF, so AA′ ∥CC′.
Therefore, $m(\measuredangle BAA') = m(\measuredangle A'AD) = m(\measuredangle CC'B)$ =x° and m($\measuredangle$ $DA'A$) = m($\measuredangle BCC'$) = m($\measuredangle C'CD$) =y°

In $\Delta CBC'$, m($\measuredangle C'BC$)=180°-x°-y° and in $\Delta ADA'$, m($\measuredangle ADA'$)= 180°-x°-y°, therefore *m($\measuredangle ADC$)=m($\measuredangle$ ABC)*, so $\measuredangle$D≡ $\measuredangle$B.

Similarly, the bisectors of the angles B and D will be B′ and D′ and the proof's approach is the same as before, so m($\measuredangle$ DAB) $= m(\measuredangle$ BCD) and $\measuredangle$A≡ $\measuredangle$C.

In consequence, ABCD is a parallelogram, due to the fact that its opposite angles are congruent.

Conclusion: As we have discovered throughout the demonstrations, regular polygons and parallelograms are balanced. However, we think that this research topic has more to offer than convex polygons. Therefore, as a team, we would like to extrapolate the subject, so as to include concave polygons. Moreover, in the future, we would be more than glad to take a shot at three-dimensional figures, as suggested by a student during the 30th annual "MATh.en.JEANS" Congress held in Iasi. We hope you found our topic interesting, because we certainly did.

Editing notes

(1) In fact, the title of this section is: if a triangle is balanced, it is equilateral (the fact that an equilateral triangle is balanced is proved §1.a).

(2) This remains true for any convex polygon whose sides have the same length.

(3) And AB<BC (in the case when two sides have equal lengths but the third one is shorter, the above proof applies).

(4) This statement is not clear. The students give a *reductio ad absurdum*: they prove that if the triangle is isosceles (and not equilateral) and balanced, then the distance h_1 from any interior point to the line BC is constant, what they mean by "M is on a parallel to BC". As the property cannot hold for every point inside the triangle, we have a contradiction.

It would have been easier to conclude directly from the first part of the proof (which still applies to the case AB≤AC and AB≤BC): the assumption "ABC is balanced" can only be satisfied if the points A' and B' are merged with C, so that the triangle ABC must be equilateral.