# ALL STONES ON THEIR WHITE SIDES

**School year:** 2017-2018
**Students:** Zuzanna Jędrzejowska, Ewa Logé, Aleksandra Wrzosek

**School :** XV Liceum Ogólnokształcące z Oddziałami Dwujęzycznymi im. Narcyzy Żmichowskiej
**Teachers :** Agata Górniak, Jolanta Otrębska,
**Researchers :** Marcin Moszyński, Uniwersytet Warszawski
**Linguistic consultations:** Ewa Drobek

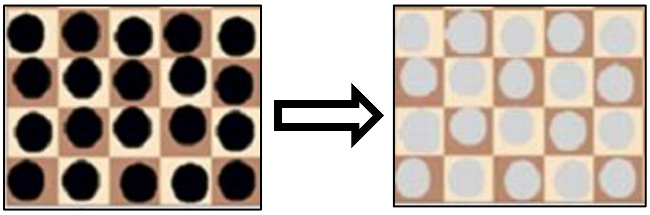
## INTRODUCTION AND RULES OF THE GAME

Our team comes from Warsaw, Poland, and we are students from Żmichowska High School. Throughout this school year (2017/2018) we were taking part in a project Erasmus+ called "Maths & Languages". Our group worked on the subject "All Stones on their white sides". We have also collaborated with our classmates Maria Kubik and Jagoda Pabich and our twin school from Souillac, France.

Consider a game with following rules:

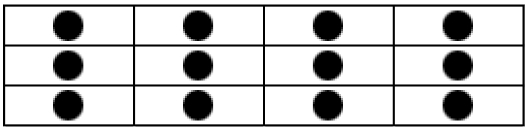
- It is a one player game
- There is a rectangular board m x n
- On each square there are stones with two sides - white and black (these are regular game pawns)
- At the beginning of the game all of the stones are on their black side (the board is all black)
- Every time a stone is chosen, stones around reverse, except for the one that had been chosen
- The objective is to obtain a completely white board

Our mathematical issue was to find winning strategies for different dimensions of the board m x n. The aim of the project was to resolve a given problem using tools we know.

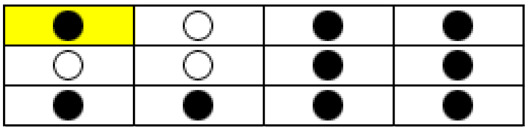


**EXAMPLE OF THE GAME**

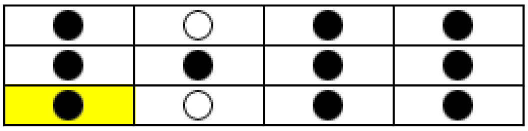
1. We start with the board all black



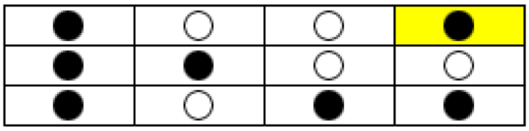
2. After having chosen the top left corner stone, three stones around it become white



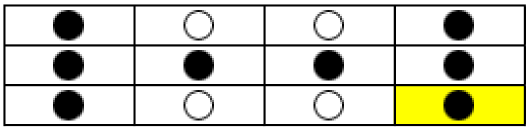
3. We chose the bottom left corner stone and three stones around it change its color



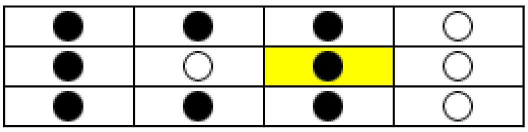
4. We chose top right corner stone…



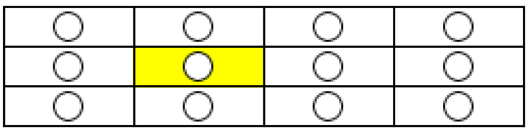
5. …bottom right corner stone…



6. …then a stone in the middle…



7. … and after having chosen another stone in the middle the board becomes all white



## FIRST DISCOVERIES

We started to play the game, to observe how the game works for different boards. Soon we concluded that:

1) Firstly, it is useless to choose the same stone two times - the stones that change colour twice return to the original colour.

2) Secondly, the order of the choices doesn't matter (1)

In conclusion: a stone changes its colour only after being chosen an odd number of times. (2)

In our first example a certain pattern appeared:

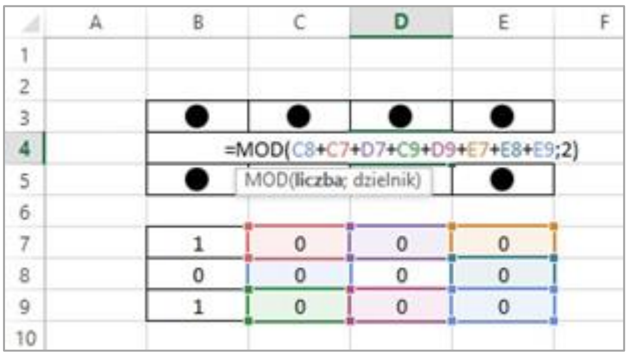
| | | | |
|---|---|---|---|
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |

The pattern of choices will be marked like that:

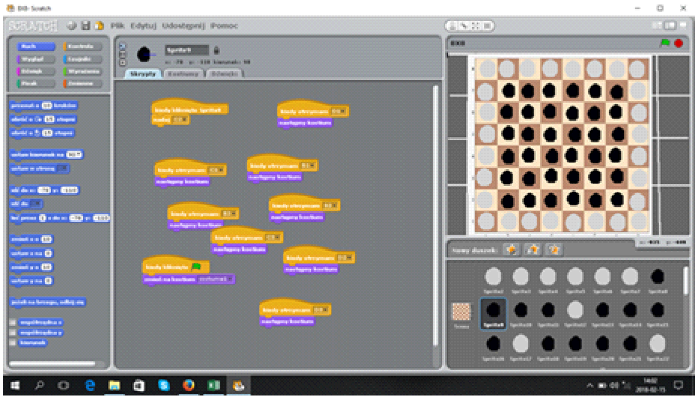
| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |

## OUR TOOLS:

We started our work with a classical pen and paper, but in order to consider multiple examples, we had to use more advanced technology. We decided that the most useful would be Excel and most suitable for this particular game, Scratch.



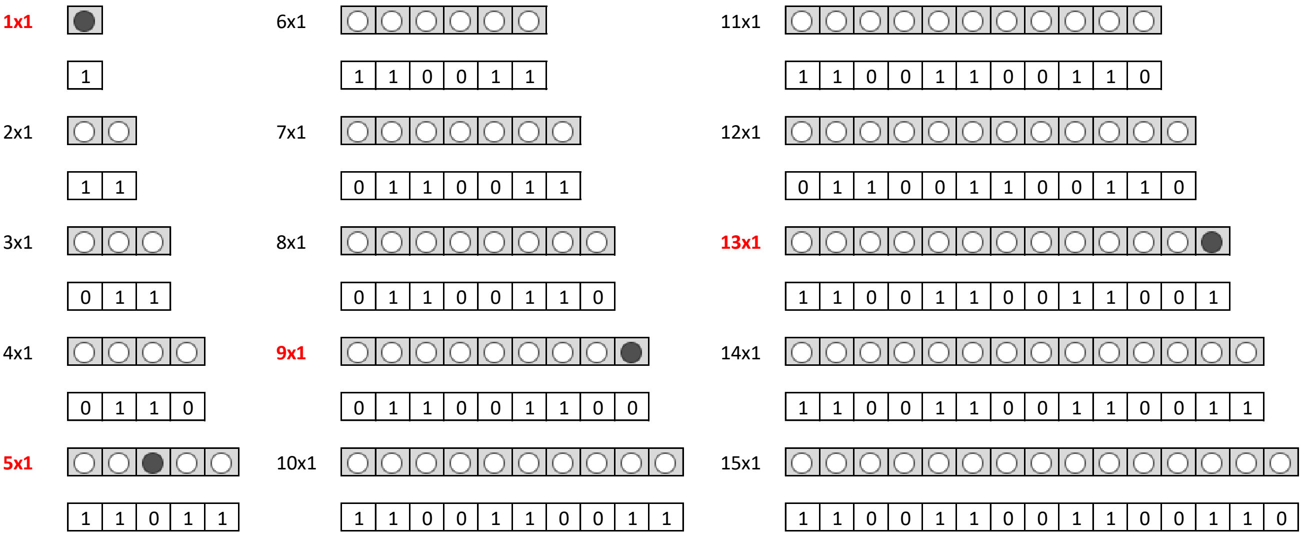

We've used an Excel's function called „MOD" in order to help us create a faster way to play our game in that program.



That’s how our game in Scratch looked like.

## FIRST OBSERVATIONS

As we started to experiment and play the game multiple times, a certain pattern appeared:




It’s visible that for some boards we couldn’t find the solution, (in the picture there’s only 1 example of a board that didn’t work, but we checked it with a lot more boards), so we started to wonder why and if we can explain it mathematically. We discovered that the secret is the divisibility by four.

## THEOREM FOR THE ONE-DIMENSIONAL BOARD

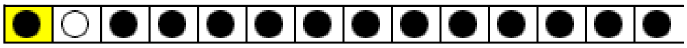
*We cannot win the game with a board* 1×n *if the remainder of the division of* n *by* 4 *is equal to* 1. (3)

Proof:

We can start with the first stone or with the second one.

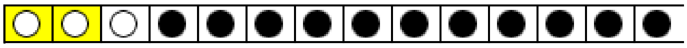
I

1. We start with the first stone



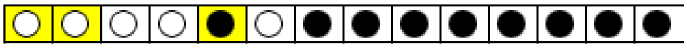
| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

2. We need to choose the second one (4)



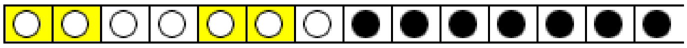
| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

3. We choose the fifth… (5)



| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

4. …and the sixth

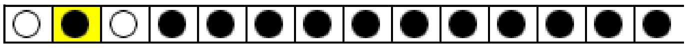


| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

As we can see a certain pattern appears: 11001100…

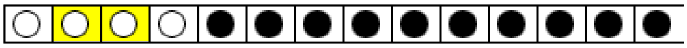
II

1. We choose the second stone



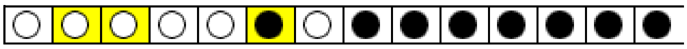
| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

2. Then the third one



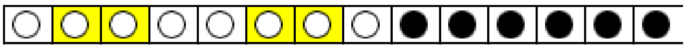
| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

3. …the sixth one



| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

4. …the seventh one



| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

We obtain a following pattern: 0110011…

We have the same situation at the end of the board so:

- We need to start/finish with 011 or 110
- We have ...00110011... in the middle of the board

In conclusion:

- We can start and finish with 11

  1100 1100 1100 ... 1100 11 → the reminder of the division by 4 is 2

- We can start and finish with 0

  0110 0110 ... 0110 → the reminder of division by 4 is 0

- We can start with 0 and finish with 1

  0110 0110 ... 0110 011 → the reminder of division by 4 is 3

- We can start with 1 and finish with 0

  1100 1100 ... 1100 110 → the reminder of division by 4 is 3

Hence, the reminders can be: 0,2,3

**OTHER BOARDS:**

Two-dimensional board: We managed to solve every board 2×n with a very easy method: choosing every field once in a random order. Works in every given case. Examples: (6)

| | |
|---|---|
| ○ | ○ |
| ○ | ○ |

| | | |
|---|---|---|
| ○ | ○ | ○ |
| ○ | ○ | ○ |

| | |
|---|---|
| 1 | 1 |
| 1 | 1 |

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 1 |

Three-dimensional boards: 3×n boards are a little bit more problematic as we didn't come up with an unequivocal formula. We think that there is a pattern very similar to the one in one dimensional boards and we suppose (7) that every board with an even „n" is solvable. Here are some examples:

| | | | |
|---|---|---|---|
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

| | | | |
|---|---|---|---|
| 1 | | | 1 |
| | 1 | 1 | |
| 1 | | | 1 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | 1 | 1 | | | |
| | 1 | 1 | | | 1 | 1 | |
| | | | 1 | 1 | | | |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | 1 | | 1 | 1 | | 1 | | | 1 |
| | 1 | 1 | | | 1 | 1 | | | 1 | 1 | |
| 1 | | | 1 | | 1 | 1 | | 1 | | | 1 |

Four-dimensional boards: We suppose (8) that it is possible to win the game in each case but we didn't prove it.

| | | | |
|---|---|---|---|
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

| | | | |
|---|---|---|---|
| 1 | 1 | | |
| 1 | 1 | | 1 |
| 1 | 1 | | 1 |
| 1 | 1 | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | 1 | | | 1 | | 1 |
| | 1 | | | | | 1 | |
| | 1 | | | | | 1 | |
| 1 | | 1 | | | 1 | | 1 |

**OVERVIEW:**

We solved the problem for 1xn boards and 2xn boards. In some cases we managed to solve the problem for 3 and 4 dimensional boards. We encourage you to play the game and look for other sequences that might appear!

**EDITING NOTES**

(1) And hence the problem is reduced to determining a set of stones that have to be chosen.

(2) Better, a stone changes its colour only if an odd number of stones around it is chosen.

(3) The proof of the theorem can be easily modified in order to show that one can win in a board 1×n if and only if the remainder of the division of n by 4 is different from 1.

(4) It should be explicitly observed that this choice, as well as the following ones, is compulsory. Otherwise, some stone never turns to white. This observation is obviously meant to answer the question whether some strategy could possibly exist when the remainder of the division of n by 4 is 1.

(5) Then we need to choose.

(6) This is a consequence of the fact that, in a 2xn board every stone has an odd number of stones around it.

(7) In Mathematics, the correct verb is ‘conjecture’.

(8) See the previous note.